HOLE PROBABILITIES OF $SU(m+1)$ GAUSSIAN RANDOM POLYNOMIALS

by

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A dissertation submitted to Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

Baltimore, Maryland

March, 2015

Abstract

In this paper, we study hole probabilities $P_{0,m}(r,N)$ of $SU(m+1)$ Gaussian random polynomials of degree N over a polydisk $(D(0,r))^m$. When $r \geq 1$, we derive asymptotic formulas and decay rate of $\log P_{0,m}(r,N)$. In one dimensional case, we also consider hole probabilities over some general open sets and compute asymptotic formulas for the generalized hole probabilities $P_{k,1}(r,N)$ over a disk $D(0,r)$.

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Secondary Reader: Joel Spruck

Acknowledgments

I would like to express my sincere gratitude to my advisor Professor Bernard Shiffman for his patience, motivation and enthusiasm. Without his guidance and persistent support this paper would not have been possible. I would like to thank Dr. Yuan Yuan for his enlightening discussions. Also I am much obliged to my ESL instructor Doris Shiffman for her help and encouragement over the past few years. Furthermore, I want to thank the faculty and staff of the Math Department in Johns Hopkins University: Dr. Jian Kong, Dr. Richard Brown, Morris Hunt, Sabrina Raymond, Charlene Poole, Christina Bannon and many others.

Dedication

This thesis work is dedicated to my parents Jinglun Zhu and Guoya Zhong. All I have and will accomplish are only possible due to their unconditional love and support.

Contents

Chapter 1

Introduction

Hole probability is the probability that some random field never vanishes over some set. The case of Gaussian random entire functions is studied by Sodin and Tsirelson:

Theorem (Sodin, Tsirelson[9] Theorem 1)**.** *Let $\psi(z) = \sum_{k=0}^{\infty} c_k \frac{z^k}{\sqrt{k!}}$, where $c_k (k \geq 0)$ are i.i.d. standard complex Gaussian random variables. Then there exist constants $C_1 \geq C_2 > 0$ such that*

$$\exp\{-C_1 r^4\} \leq Prob\Big\{0 \notin \psi\big(D(0,r)\big)\Big\} \leq \exp\{-C_2 r^4\}.$$

Moreover, an estimate of multivariable version can be found in [10] Theorem 1.2. In [7], the authors consider the case of Gaussian random sections: let M be a compact Kähler manifold with complex dimension m and $(L,h) \to M$ be a positive holomorphic line bundle. γ_N denotes the Gaussian probability measure on $H^0(M, L^N)$ induced by the fiberwised inner product h^N and the polarized volume form $dV_M = \frac{\omega_h^m}{m!} = \frac{1}{m!}(\frac{\sqrt{-1}}{2\pi}\Theta_h)^m$, where Θ_h is the Chern curvature tensor of (L,h).

Theorem (Shiffman, Zelditch, Zrebiec[7] Theorem 1.4)**.** *For any nonempty open set $U \subset M$, if there exists $s \in H^0(M,L)$ such that s does not vanish on $\bar{U}$. Then there exist constants $C_1 \geq C_2 > 0$ such that for $N \gg 1$,*

$$\exp\{-C_1 N^{m+1}\} \leq \gamma_N\{s_N \in H^0(M,L^N) : 0 \notin s_N(U)\} \leq \exp\{-C_2 N^{m+1}\}.$$

Therefore, it is natural to ask: can we find sharp constants C_1, C_2 in the above two theorems and furthermore, is it possible to obtain an asymptotic formula and a decay rate for the hole probability? Using Cauchy's integral estimates, Nishry answers this question in the random entire function case:

Theorem (Nishry[4] Theorem 1)**.** *Let* $\psi(z) = \sum_{k=0}^{\infty} c_k \frac{z^k}{\sqrt{k!}}$, *where* $c_k (k \geq 0)$ *are i.i.d. standard complex Gaussian random variables. Then*

$$Prob\Big\{0 \notin \psi\big(D(0,r)\big)\Big\} = \exp\{-\frac{e^2}{2} r^4 + O(r^{\frac{18}{5}})\}.$$

An analogous result for Gaussian random power series is obtained in [5] Corollary 3. This inspires us that for those line bundles with polynomial sections, maybe it is possible to find an asymptotic formula for the hole probability.

If $P_{0,m}(r,N)$ denotes the hole probability of $SU(m+1)$ Gaussian random polynomials over the polydisk $\big(D(0,r)\big)^m$, $d_m x$ is the Lebesgue measure on $\mathbb{R}^m$ and

$$E_r(x) := 2\sum_{i=1}^m x_i \log r - \big[\sum_{i=1}^m x_i \log x_i + (1 - \sum_{i=1}^m x_i) \log (1 - \sum_{i=1}^m x_i)\big]$$

is a continuous function defined over the standard simplex $\Sigma_m := \{x = (x_1, \ldots, x_m) \in \mathbb{R}^{m+} : \sum_{i=1}^m x_i \leq 1\}$(here we adopt the convention that $0 \log 0 = 0$), we have the following results:

Theorem 1.1. *For* $r \geq 1$,

$$\log P_{0,m}(r,N) = -N^{m+1} \int_{\Sigma_m} E_r(x)\ d_m x + o(N^{m+1}),$$

where

$$\int_{\Sigma_m} E_r(x)\ d_m x = \frac{2m \log r}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k}.$$

Theorem 1.2. *For* $r > 0$,

$$\log P_{0,m}(r,N) \geq -N^{m+1} \int_{x \in \Sigma_m:\ E_r(x) \geq 0} E_r(x)\ d_m x + o(N^{m+1}),$$
$$\log P_{0,m}(r,N) \leq -N^{m+1} \int_{x \in \mathbb{R}^{m+}:\ \sum_{i=1}^m x_i \leq \alpha_0} E_r(x)\ d_m x + o(N^{m+1}),$$

where

$$\alpha_0 = \alpha_0(r,m) = \begin{cases} 1 & \textit{if } 2\log r + \sum_{k=2}^m \frac{1}{k} \geq 0, \\ \textit{the nonzero root of } (2 \log r + \sum_{k=2}^m \frac{1}{k})\alpha = \alpha \log \alpha + (1-\alpha) \log (1-\alpha) & \textit{if } 2\log r + \sum_{k=2}^m \frac{1}{k} < 0. \end{cases}$$

Here when $m = 1$*, we take* $\sum_{k=2}^{m} \frac{1}{k} = 0$.

Remark 1.3. *Theorem 1.1 can be derived from Theorem 1.2 as when* $r \geq 1$*,* $\{x \in \Sigma_m : E_r(x) \geq 0\} = \Sigma_m$ *and* $\alpha_0(r, m) = 1$*. In fact we could have proved this general case directly. But the idea of the proof would turn out to be extremely difficult to follow.*

Corollary 1.4. *In the case of* $m = 1$*, the asymptotic formula for the logarithm of the hole probability over a disk exists for all* $r > 0$*:*

$$\log P_{0,1}(r, N) = -N^2 \int_0^{\alpha_0} E_r(x)\ dx + o(N^2),$$

here

$$\int_0^{\alpha_0} E_r(x)\ dx = \tfrac{1}{2}\alpha_0(2\log r + 1 - \log \alpha_0),$$

and $\alpha_0 = \alpha_0(r, 1) \in (0, 1]$ *is given in Theorem 1.2.*

Because of the simplicity of one dimensional case, we can obtain more about the hole probability of $SU(2)$ Gaussian random polynomials:

Theorem 1.5. *If* $U \subset \mathbb{C}$ *is a bounded simply connected domain containing* 0 *and* ∂U *is a Jordan curve. Let* $\phi : D(0, 1) \to U$ *be a biholomorphism given by the Riemmann mapping theorem such that* $\phi(0) = 0$*(thus* ϕ *is unique up to the composition of a unitary transformation of* $\mathbb{C}$*). Then the hole probability* $P_{0,1}(U, N)$ *of* $SU(2)$ *Gaussian random polynomials of degree* N *over* U *satisfies*

$$\log P_{0,1}(U, N) \leq -(\log|\phi'(0)| + \tfrac{1}{2})N^2 + o(N^2).$$

Also in dimension one, it makes sense to study the number of zeros in some set. So let a generalized hole probability $P_{k,1}(r, N)$ be the probability that an $SU(2)$ Gaussian random polynomial of degree N has no more than k zeros in $D(0, r)$, then the following theorem shows that asymptotic formula of $\log P_{k,1}(r, N)$ exists:

Theorem 1.6. *For all* $k \geq 0$ *and* $r > 0$*:*

$$\log P_{k,1}(r, N) = -\tfrac{1}{2}\alpha_0(2\log r + 1 - \log \alpha_0)N^2 + o(N^2),$$

where $\alpha_0 = \alpha_0(r, 1) \in (0, 1]$ *is given in Theorem 1.2.*

We should remark here that in all the cases we consider, the event that some Gaussian random polynomial has zeros on the boundary of some open set is a null set, i.e. of zero probability. Therefore we do not distinguish between the (generalized) hole probability over an open set and that over its closure.

Chapter 2

Background

We review in this chapter some background on $SU(m+1)$ Gaussian random polynomials and the definition of our probability measures. Before that, let's define two lexicographically ordered sets that will be consistently used as index sets throughout this paper.

Definition 2.1.

$$\Gamma_{m,N} := \{J = (j_1, \ldots, j_m) \in [0,N]^m \cap \mathbb{Z}^m : \ 0 \le j_1 \le \cdots \le j_m \le N\},$$

$$\Lambda_{m,N} := \{K = (k_1, \ldots, k_m) \in [0,N]^m \cap \mathbb{Z}^m : \ |K| = k_1 + \cdots + k_m \le N\}.$$

It is not difficult to show that $|\Gamma_{m,N}| = |\Lambda_{m,N}| = \binom{N+m}{m}$.

The tautological line bundle $\mathcal{O}(-1)$ over the complex projective space $\mathbb{CP}^m$ is a holomorphic line bundle with fibers

$$\mathcal{O}(-1)_{[x]} = \mathbb{C} \cdot x, \qquad \text{for all } [x] = [x_0 : \cdots : x_m] \in \mathbb{CP}^m.$$

Its dual bundle, denoted by $\mathcal{O}(1)$, is called the hyperplane section bundle since $\mathcal{O}(1) = \mathcal{O}(H)$ where the divisor

$$H = \{[x] \in \mathbb{CP}^m : \ x_0 = 0\}$$

is a hyperplane in $\mathbb{CP}^m$. $H^0(\mathbb{CP}^m, \mathcal{O}(N))$, the space of holomorphic sections of the tensor bundle $\mathcal{O}(N) = \mathcal{O}(1)^{\otimes N}$, is isomorphic to ${}^h\mathcal{P}^N_{m+1}$, the space of $(m+1)$–variable homogenous polynomials of degree N. The Fubini-Study metric h_{FS} on $\mathcal{O}(1)$ can be described in the following way: over the open subset

$$U_0 = \Big\{[x] = [x_0 : \cdots : x_m] \in \mathbb{CP}^m : \ x_0 \neq 0\Big\} \subset \mathbb{CP}^m,$$

we have a local frame of $\mathcal{O}(1)$

$$e([x]) = x_0.$$

Set

$$\|e([x])\|^2_{h_{\mathrm{FS}}} = \frac{|x_0|^2}{\sum\limits_{i=0}^{m} |x_i|^2} = \frac{|x_0|^2}{\|x\|^2},$$

which is independent of the choice of representative x of $[x]$. In terms of affine coordinate

$$z = (z_1, \dots, z_m) = \Big(\frac{x_1}{x_0}, \dots, \frac{x_m}{x_0}\Big)$$

over U_0,

$$\|e(z)\|^2_{h_{\mathrm{FS}}} = (1 + \|z\|^2)^{-1} = \Big(1 + \sum_{i=1}^{m} |z_i|^2\Big)^{-1},$$

which defines a metric with positive Chern curvature form

$$\omega_{\mathrm{FS}} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|e(z)\|^2_{h_{\mathrm{FS}}} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(1 + |z_1|^2 + \cdots + |z_m|^2\right).$$

This induces a metric h^N_{FS} on the line bundle $\mathcal{O}(N)$ so that

$$\|e^{\otimes N}(z)\|^2_{h^N_{\mathrm{FS}}} = (1 + \|z\|^2)^{-N}.$$

With the frame $e^{\otimes N}$ over U_0, for any $s \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$ which is represented by $p(x_0, \dots, x_m) \in {}^h\mathcal{P}^N_{m+1}$, we have

$$p(x_0, \dots, x_m) = \frac{p(x_0, \dots, x_m)}{x_0^N} e^{\otimes N}([x]) = p(1, z_1, \dots, z_m) e^{\otimes N}([x]),$$

which implies that all the elements in $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ can be viewed over U_0 as polynomials in $(z_1, \dots, z_m)$ of degree at most N.

Since ω_{FS} is positive over $\mathbb{CP}^m$, we may take it as a polarized metric form on $\mathbb{CP}^m$ and the associated volume form is $dV = \frac{\omega_{\mathrm{FS}}^m}{m!}$. Thus, the metric h_{FS}^N together with the volume form dV induce a Hermitian inner product on the space of holomorphic sections $H^0(\mathbb{CP}^m, \mathcal{O}(N))$: for all $s_1, s_2 \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$,

$$\langle\!\langle s_1, s_2 \rangle\!\rangle := \int_{\mathbb{CP}^m} \langle s_1, s_2 \rangle_{h_{\mathrm{FS}}^N} \, dV.$$

With this inner product, there is an orthonormal basis $\{S_K^N\}_{K=(k_1,\dots,k_m)\in\Lambda_{m,N}}$, given in local affine coordinates $(z_1, \dots, z_m)$ over U_0 by

$$S_K^N(z) = \sqrt{(N+1)\cdots(N+m)} \sqrt{\binom{N}{K}} z^K,$$

where we adopt the notations

$$\binom{N}{K} = \frac{N!}{(N-|K|)! k_1! \cdots k_m!}, \quad z^K := z_1^{k_1} \cdots z_m^{k_m}.$$

Thus $H^0(\mathbb{CP}^m, \mathcal{O}(N)) = \left\{ s_N = \sum_{K\in\Lambda_{m,N}} c_K S_K^N : \, c = (c_K)_{K\in\Lambda_{m,N}} \in \mathbb{C}^{\binom{N+m}{m}} \right\}$. Endow $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ with the Gaussian probability measure γ_N defined by

$$d\gamma_N(s_N) := \pi^{-\binom{N+m}{m}} e^{-\|c\|^2} \, d_{2\binom{N+m}{m}} c,$$

where $\|c\|^2 = \sum_{K\in\Lambda_{m,N}} |c_K|^2$ and $d_{2\binom{N+m}{m}} c$ denotes the $2\binom{N+m}{m}$–dimensional Lebesgue measure. γ_N is characterized by the property that $\{c_K\}_{K\in\Lambda_{m,N}}$ are independent and identically distributed(i.i.d.) standard complex Gaussian random variables. Then $(H^0(\mathbb{CP}^m, \mathcal{O}(N)), \ \gamma_N)$ is called the ensemble of $SU(m+1)$ Gaussian random polynomials of degree N as the random element s_N is distributional invariant under $SU(m+1)$ transformations of $\mathbb{CP}^m$. Its hole probability over the polydisk $(D(0,r))^m \subset \mathbb{C}^m$ is

$$\begin{aligned} P_{0,m}(r,N) &= \gamma_N \left\{ s_N \in H^0(\mathbb{CP}^m, \mathcal{O}(N)) : \ 0 \notin s_N \left((\bar{D}(0,r))^m \right) \right\} \\ &= \pi^{-\binom{N+m}{m}} \int_{c\in\mathbb{C}^{\binom{N+m}{m}} : \ 0\notin s_N\left((\bar{D}(0,r))^m\right)} e^{-\|c\|^2} \, d_{2\binom{N+m}{m}} c \\ &= \pi^{-\binom{N+m}{m}} \int_{c\in\mathbb{C}^{\binom{N+m}{m}} : \ 0\notin \tilde{s}_N\left((\bar{D}(0,r))^m\right)} e^{-\|c\|^2} \, d_{2\binom{N+m}{m}} c, \end{aligned}$$

where $\tilde{s}_N(z) = \sum_{K\in\Lambda_{m,N}} c_K \sqrt{\binom{N}{K}} z^K$. Thereafter, when considering hole probability, we work on $\tilde{s}_N$

instead of s_N for simplicity.

Chapter 3

Preliminaries

Definition 3.1. $Q_{r,m}(N) := \sum_{K \in \Lambda_{m,N}} \log\Big[\binom{N}{K} r^{2|K|}\Big]$

Lemma 3.2.

$$Q_{r,m}(N) = N^{m+1} \int_{\Sigma_m} E_r(x)\ d_m x + o(N^{m+1}) = \Big[\frac{2m \log r}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k}\Big] N^{m+1} + o(N^{m+1}).$$

Proof. We can prove inductively that for $k \geq 1$,

$$\Big(\frac{k}{e}\Big)^k \leq k! \leq \frac{k^{k+1}}{e^{k-1}},$$

or equivalently,

$$k \log k - k \leq \log k! \leq (k+1) \log k - (k-1). \tag{3.1}$$

Hence we have

$$-(k+1) \log N + (k-1) \leq k \log \frac{k}{N} - \log k! \leq -k \log N + k \text{ for } 0 \leq k \leq N. \tag{3.2}$$

For all $K = (k_1, \ldots, k_m) \in \Lambda_{m,N}$,

$$\log\Big[\binom{N}{K} r^{2|K|}\Big] - N E_r\Big(\frac{K}{N}\Big) = \log N! + \sum_{i=1}^{m} \Big(k_i \log \frac{k_i}{N} - \log k_i!\Big) + \Big[(N - |K|) \log \frac{N - |K|}{N} - \log (N - |K|)!\Big],$$

Applying (3.1) and (3.2), we then get

$$\log\Big[\binom{N}{K}r^{2|K|}\Big] - NE_r(\frac{K}{N}) \geq (N\log N - N) - (N+m+1)\log N + (N-m-1) = -(m+1)(\log N + 1),$$
$$\log\Big[\binom{N}{K}r^{2|K|}\Big] - NE_r(\frac{K}{N}) \leq [(N+1)\log N - (N-1)] - N\log N + N = \log N + 1.$$

Hence for all $K \in \Lambda_{m,N}$,

$$\Big|\log\Big[\binom{N}{K}r^{2|K|}\Big] - NE_r(\frac{K}{N})\Big| \leq (m+1)(\log N + 1),$$

so

$$\begin{aligned} \Big|Q_{r,m}(N) - N\sum_{K\in\Lambda_{m,N}} E_r(\frac{K}{N})\Big| &\leq \sum_{K\in\Lambda_{m,N}} \Big|\log\Big[\binom{N}{K}r^{2|K|}\Big] - NE_r(\frac{K}{N})\Big| \\ &\leq (m+1)(\log N+1)\binom{N+m}{m} = o(N^{m+1}). \end{aligned} \tag{3.3}$$

Take

$$\mathring{\Lambda}_{m,N} := \{K \in \Lambda_{m,N} : \ k_i \geq 1 \text{ for } 1 \leq i \leq m \text{ and } |K| \leq N-m-1\} \subset \Lambda_{m,N}$$

and

$$\mathring{\Sigma}_m(N) := \bigcup_{K\in\mathring{\Lambda}_{m,N}} [\frac{k_1}{N}, \frac{k_1+1}{N}] \times \cdots \times [\frac{k_m}{N}, \frac{k_m+1}{N}] \subset \Sigma_m.$$

Then

$$\begin{aligned} |\mathring{\Lambda}_{m,N}| &= \binom{N-m-1}{m}, \\ |\Lambda_{m,N} \smallsetminus \mathring{\Lambda}_{m,N}| &= \binom{N+m}{m} - \binom{N-m-1}{m} = O(N^{m-1}), \\ \mathrm{Vol}_{\mathbb{R}^m}(\Sigma_m \smallsetminus \mathring{\Sigma}_m(N)) &= \frac{1}{m!} - N^{-m}\binom{N-m-1}{m} = O(N^{-1}). \end{aligned}$$

Over Σ_m we have

$$|E_r| \leq 2|\log r| + \frac{m+1}{e} = O(1),$$

hence

$$\left|N \sum_{K \in \Lambda_{m,N}} E_r\left(\frac{K}{N}\right) - N \sum_{K \in \mathring{\Lambda}_{m,N}} E_r\left(\frac{K}{N}\right)\right| \le N|\Lambda_{m,N} \setminus \mathring{\Lambda}_{m,N}| \sup_{\Sigma_m} |E_r| = O(N^m). \tag{3.4}$$

As

$$\sup_{\mathring{\Sigma}_m(N)} \|\nabla E_r\| \le O(\log N),$$

then

$$\begin{aligned}
&\left|N \sum_{K \in \mathring{\Lambda}_{m,N}} E_r\left(\frac{K}{N}\right) - N^{m+1} \int_{\mathring{\Sigma}_m(N)} E_r(x)\ d_m x\right| \\
\le & N^{m+1} \sum_{K \in \mathring{\Lambda}_{m,N}} \int_{[\frac{k_1}{N}, \frac{k_1+1}{N}] \times \cdots \times [\frac{k_m}{N}, \frac{k_m+1}{N}]} \left|E_r\left(\frac{K}{N}\right) - E_r(x)\right| \ d_m x \\
\le & N^{m+1} \binom{N-m-1}{m} N^{-m} O(\log N) O(N^{-1}) \\
= & O(N^m \log N).
\end{aligned} \tag{3.5}$$

Moreover,

$$\left|N^{m+1} \int_{\mathring{\Sigma}_m(N)} E_r(x)\ d_m x - N^{m+1} \int_{\Sigma_m} E_r(x)\ d_m x\right| \le N^{m+1} \sup_{\Sigma_m} |E_r| \mathrm{Vol}_{\mathbb{R}^m}(\Sigma_m \setminus \mathring{\Sigma}_m(N)) = O(N^m). \tag{3.6}$$

Combining (3.3)~(3.6), we thus obtain

$$\begin{aligned}
Q_{r,m}(N) &= N^{m+1} \int_{\Sigma_m} E_r(x)\ d_m x + o(N^{m+1}) \\
&= N^{m+1} \int_{\Sigma_m} 2 \sum_{i=1}^m x_i \log r - \Big[\sum_{i=1}^m x_i \log x_i + (1 - \sum_{i=1}^m x_i) \log (1 - \sum_{i=1}^m x_i)\Big]\ d_m x + o(N^{m+1}) \\
&= N^{m+1} \Big[2m \log r \int_{\Sigma_m} x_1\ d_m x - (m+1) \int_{\Sigma_m} x_1 \log x_1\ d_m x\Big] + o(N^{m+1}) \\
&= \Big[\frac{2m \log r}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k}\Big] N^{m+1} + o(N^{m+1}).
\end{aligned}$$

□

Remark 3.3. *The scaled lattice* $\frac{1}{N}\Lambda_{m,N} \subset \mathbb{R}^m$ *tends to* Σ_m*. Hence Lemma 3.2 is in fact converting a Riemann sum into a Riemann integral and estimating the error. Such procedures will appear several times in this paper.*

Remark 3.4. *The function* $E_r(x)$ *in the above lemma can also be written as* $E_r(x) = -b_{\{x\}}(z_r) +$

$\log(1+\|z_r\|^2)$, *where* $z_r = (r,\ldots,r) \in \mathbb{R}^m$ *and* $b_{\{x\}}$ *is the exponential decay rate of the expected mass density of random* L^2 *normalized polynomials with some prescribed Newton polytope(see Theorem 1.2 and (78) in* [6]*).*

Let $\xi = (\xi_1,\ldots,\xi_m)$, where for $1 \le i \le m$, $\xi_i = (\xi_{i,0},\ldots,\xi_{i,N}) \in \mathbb{C}^{N+1}$.

Definition 3.5. *$W_{m,N}(\xi)$ is the $\binom{N+m}{m} \times \binom{N+m}{m}$ matrix with rows indexed by $\Gamma_{m,N}$ and columns indexed by $\Lambda_{m,N}$, such that for all $J = (j_1,\ldots,j_m) \in \Gamma_{m,N}$, $K = (k_1,\ldots,k_m) \in \Lambda_{m,N}$, the (J,K)-entry of $W_{m,N}(\xi)$ is $\xi_J^K = \xi_{1,j_1}^{k_1}\cdots\xi_{m,j_m}^{k_m}$.*

Next lemma gives the formula for a "Vandermonde type" determinant.

Lemma 3.6. $|\det W_{m,N}(\xi)| = \prod_{i=1}^m \prod_{0\le j<k\le N} |\xi_{i,j} - \xi_{i,k}|^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}}$.

Proof. For all $1 \le i \le m$ and $0 \le j < k \le N$, the rows of $W_{m,N}(\xi)$ involving $\xi_{i,j}$ correspond to the set

$$\Gamma_{m,N}^{i,j} = \{(j_1,\ldots,j_m) \in \Gamma_{m,N} : j_i = j\}$$

while those rows involving $\xi_{i,k}$ correspond to the set

$$\Gamma_{m,N}^{i,k} = \{(j_1,\ldots,j_m) \in \Gamma_{m,N} : j_i = k\}. \tag{3.7}$$

Let

$$\tilde{\Gamma}_{m,N}^{i,j} = \{(j_1,\ldots,\hat{j}_i,\ldots,j_m) \in [0,N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \le j_1 \le \cdots \le j_{i-1} \le j \le j_{i+1} \le \cdots \le j_m \le N\},$$
$$\tilde{\Gamma}_{m,N}^{i,k} = \{(j_1,\ldots,\hat{j}_i,\ldots,j_m) \in [0,N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \le j_1 \le \cdots \le j_{i-1} \le k \le j_{i+1} \le \cdots \le j_m \le N\},$$

then

$$|\Gamma_{m,N}^{i,j}| = |\tilde{\Gamma}_{m,N}^{i,j}| = \binom{j+i-1}{i-1}\binom{N-j+m-i}{m-i},$$
$$|\Gamma_{m,N}^{i,k}| = |\tilde{\Gamma}_{m,N}^{i,k}| = \binom{k+i-1}{i-1}\binom{N-k+m-i}{m-i}.$$

Since for any $1 \le i \le m$,

$$\Gamma_{m,N} = \bigsqcup_{k=0}^N \Gamma_{m,N}^{i,k},$$

we thus have the equality

$$\sum_{k=0}^{N} \binom{k+i-1}{i-1} \binom{N-k+m-i}{m-i} = \binom{N+m}{m}. \tag{3.8}$$

Note that

$$\tilde{\Gamma}^{i,j}_{m,N} \cap \tilde{\Gamma}^{i,k}_{m,N} = \{(j_1, \ldots, \hat{j}_i, \ldots, j_m) \in [0,N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \le j_1 \le \cdots \le j_{i-1} \le j < k \le j_{i+1} \le \cdots \le j_m \le N\}$$

and

$$|\tilde{\Gamma}^{i,j}_{m,N} \cap \tilde{\Gamma}^{i,k}_{m,N}| = \binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i},$$

which means that there are $\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}$ pairs of rows, within each pair the only difference between two rows is replacing $\xi_{i,j}$ by $\xi_{i,k}$. Therefore, for all $1 \le i \le m$ and $0 \le j < k \le N$,

$$(\xi_{i,j} - \xi_{i,k})^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}} | \det W_{m,N}(\xi),$$

and thus

$$G_{m,N}(\xi) | \det W_{m,N}(\xi), \tag{3.9}$$

where

$$G_{m,N}(\xi) \coloneqq \prod_{i=1}^{m} \prod_{0 \le j < k \le N} (\xi_{i,j} - \xi_{i,k})^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}}.$$

Furthermore, for all $1 \le i \le m$,

$$\begin{aligned}
\deg_{\xi_i} G_{m,N}(\xi) &= \sum_{0 \le j < k \le N} \binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i} \\
&= \sum_{k=1}^{N} \Big[\sum_{j=0}^{k-1} \binom{j+i-1}{i-1} \Big] \binom{N-k+m-i}{m-i} \\
&= \sum_{k=1}^{N} \binom{k-1+i}{i} \binom{N-k+m-i}{m-i} \\
&= \sum_{k-1=0}^{N-1} \binom{(k-1)+(i+1)-1}{(i+1)-1} \binom{(N-1)-(k-1)+(m+1)-(i+1)}{(m+1)-(i+1)} \\
&= \binom{(N-1)+(m+1)}{m+1} \\
&= \binom{N+m}{m+1},
\end{aligned} \tag{3.10}$$

where the second to last equality is due to (3.8). On the other hand, for all $1 \le i \le m$ and $1 \le k \le N$, the number of K's in $\Lambda_{m,N}$ with $k_i = k$ is $\binom{N-k+m-1}{m-1}$, hences

$$\begin{aligned}
\deg_{\xi_i} \det W_{m,N}(\xi) &= \sum_{k=1}^{N} k \binom{N-k+m-1}{m-1} \\
&= \binom{N+m}{m+1},
\end{aligned}$$

where the second equality is the special case $i = 1$ in (3.10). Therefore, for all $1 \le i \le m$,

$$\deg_{\xi_i} \det W_{m,N}(\xi) = \deg_{\xi_i} G_{m,N}(\xi). \tag{3.11}$$

$$(3.9) \text{ and } (3.11) \Rightarrow \det W_{m,N}(\xi) = C_{m,N} G_{m,N} = C_{m,N} \prod_{i=1}^{m} \prod_{0 \le j < k \le N} (\xi_{i,j} - \xi_{i,k})^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}},$$

where $C_{m,N}$ is a constant depending only on m and N. Consider the monomial

$$g_{m,N}(\xi) := \prod_{i=1}^{m} \prod_{k=1}^{N} \xi_{i,k}^{\sum_{j=0}^{k-1} \binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}} = \prod_{i=1}^{m} \prod_{k=1}^{N} \xi_{i,k}^{\binom{k+i-1}{i}\binom{N-k+m-i}{m-i}},$$

then

$$G_{m,N}(\xi) = \pm g_{m,N}(\xi) + \dots$$

In the appendix, we show that the coefficient of $g_{m,N}$ in the expansion of $\det W_{m,N}(\xi)$ equals 1, and therefore $C_{m,N} = \pm 1$. $\square$

Chapter 4

Proof of Theorem 1.1

To prove Theorem 1.1, it suffices to prove separately the lower bound

$$\log P_{0,m}(r,N) \geq -N^{m+1}\int_{\Sigma_m} E_r(x)d_mx + o(N^{m+1})$$

and upper bound

$$\log P_{0,m}(r,N) \leq -N^{m+1}\int_{\Sigma_m} E_r(x)d_mx + o(N^{m+1}).$$

4.1 Lower bound

Proof of the lower bound in Theorem 1.1. Recall that $\tilde{s}_N(z) = \sum_{K\in\Lambda_{m,N}} c_K\sqrt{\binom{N}{K}}z^K$. Hence,

$$|\tilde{s}_N(z)| \geq |c_{(0,\dots,0)}| - \sum_{K\in\Lambda_{m,N}\backslash\{(0,\dots,0)\}} |c_K|\sqrt{\binom{N}{K}}r^{|K|}, \quad \text{for all } z = (z_1,\dots,z_m) \in (\bar{D}(0,r))^m. \tag{4.1}$$

Consider the event $\Omega_{r,m,N}$:

(i) $|c_{(0,\dots,0)}| \geq \sqrt{N}$,

(ii) $|c_K| \leq \dfrac{1}{2\sqrt{N}\sqrt{\binom{N}{K}}r^{|K|}\binom{|K|+m-1}{m-1}}$, $K \in \Lambda_{m,N}\backslash\{(0,\dots,0)\}$.

Then if $\Omega_{r,m,N}$ occurs, by (4.1), we have that for all $z = (z_1, \ldots, z_m) \in (\bar{D}(0,r))^m$,

$$
\begin{aligned}
|\tilde{s}_N(z)| &\geq \sqrt{N} - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\ldots,0)\}} \frac{\sqrt{\binom{N}{K}} r^{|K|}}{2\sqrt{N}\sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}} \\
&= \sqrt{N} - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\ldots,0)\}} \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}} \\
&= \sqrt{N} - \sum_{k=1}^{N} \frac{1}{2\sqrt{N}} \\
&= \tfrac{1}{2}\sqrt{N} > 0,
\end{aligned}
$$

hence

$$
P_{0,m}(r,N) \geq \gamma_N(\Omega_{r,m,N}) = \gamma_N(|c_{(0,\ldots,0)}| \geq \sqrt{N}) \prod_{K \in \Lambda_{m,N} \setminus \{(0,\ldots,0)\}} \gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}}\right),
$$

where $\gamma_N(|c_{(0,\ldots,0)}| \geq \sqrt{N}) = e^{-N}$. Recall that for $K \in \Lambda_{m,N} \setminus \{(0,\ldots,0)\}$, the standard complex Gaussian random variables c_K satisfy $\gamma_N(|c_K| \leq a) \geq \frac{1}{2}a^2$ whenever $a \leq 1$. Since $\frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}} \leq 1$ if $r \geq 1$, we thus have

$$
\gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}}\right) \geq \frac{1}{2}\left[\frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}}\right]^2 = \frac{1}{8N\binom{N}{K} r^{2|K|} \binom{|K|+m-1}{m-1}^2},
$$

and

$$
\log P_{0,m}(r,N) \geq -N - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\ldots,0)\}} \left\{ \log 8 + \log N + 2\log\binom{|K|+m-1}{m-1} + \log\left[\binom{N}{K} r^{2|K|}\right] \right\}.
$$

Since

$$
\log\binom{|K|+m-1}{m-1} \leq \log\binom{N+m-1}{m-1} = O(\log N),
$$

hence

$$
\sum_{K \in \Lambda_{m,N} \setminus \{(0,\ldots,0)\}} \left[\log 8 + \log N + 2\log\binom{|K|+m-1}{m-1} \right] = \binom{N+m}{m} O(\log N) = o(N^{m+1}).
$$

Therefore,

$$\log P_{0,m}(r,N) \geq -\sum_{K\in\Lambda_{m,N}\setminus\{(0,\ldots,0)\}} \log\left[\binom{N}{K} r^{2|K|}\right] + o(N^{m+1})$$
$$= -Q_{r,m}(N) + o(N^{m+1}) = -N^{m+1}\int_{\Sigma_m} E_r(x)\ d_m x + o(N^{m+1}).$$

□

4.2 Upper bound

Let $\delta > 0$ be small, $\kappa = 1 - \sqrt{\delta}$. We shall first treat δ as a small positive constant and at the end we will let $\delta \to 0+$. For the sake of clarity, all the constants C, capital O and little o terms listed throughout this paper will not depend on δ unless otherwise stated.

Definition 4.1. $z_j(N) \coloneqq \kappa r e^{2\pi\sqrt{-1}\frac{j}{N+1}}$, *for* $0 \leq j \leq N$.

For all $p \in \mathbb{Z}^+$, by division with remainder, $N + 1 = q(N)p + l(N)$, where $q(N) \in \mathbb{Z}$, $q(N) \geq 0$ and $0 \leq l(N) < p$. For convenience, we drop the dependence on N when there is no confusion. For all $1 \leq i \leq m$, assign the values of $\xi_i = (\xi_{i,0}, \ldots, \xi_{i,N})$ by means of the table below:

$\xi_{i,0} = z_0$	$\cdots$	$\xi_{i,(q-1)p} = z_{q-1}$	$\xi_{i,qp} = z_q$
$\xi_{i,1} = z_{q+1}$	$\cdots$	$\xi_{i,(q-1)p+1} = z_{(q+1)+(q-1)}$	$\xi_{i,qp+1} = z_{(q+1)+q}$
....	$\cdots$	....	....
$\xi_{i,l-1} = z_{(l-1)(q+1)}$	$\cdots$	$\xi_{i,(q-1)p+(l-1)} = z_{(l-1)(q+1)+(q-1)}$	$\xi_{i,qp+(l-1)} = z_{(l-1)(q+1)+q}$
$\xi_{i,l} = z_{l(q+1)}$	$\cdots$	$\xi_{i,(q-1)p+l} = z_{l(q+1)+(q-1)}$	
....	$\cdots$	....	
$\xi_{i,p-1} = z_{l(q+1)+(p-1-l)q}$	$\cdots$	$\xi_{i,(q-1)p+(p-1)} = z_{l(q+1)+(p-1-l)q+(q-1)}$	

(4.2)

Intuitively, table (4.2) gives a way to choose points $\xi_{i,j}(j = 0, 1, \ldots)$ one after another on the circle of radius κr that the arguments of each two consecutive points differ approximately by $\frac{2\pi}{p}$. Denote the permutation of $N + 1$ indices $\{0, \ldots, N\}$ indicated in table (4.2) by τ, i.e. $z_j = \xi_{i,\tau(j)}$ for

$0 \le j \le N$ and $1 \le i \le m$. Denote

$$I_0 = \{0, \dots, q\},\ a_0 = 0,$$

$$I_1 = \{q+1, \dots, (q+1)+q\},\ a_1 = q+1,$$

$$\dots$$

$$I_{l-1} = \{(l-1)(q+1), \dots, (l-1)(q+1)+q\},\ a_{l-1} = (l-1)(q+1),$$

$$I_l = \{l(q+1), \dots, l(q+1)+(q-1)\},\ a_l = l(q+1),$$

$$\dots$$

$$I_{p-1} = \{l(q+1)+(p-1-l)q, \dots, l(q+1)+(p-1-l)q+(q-1)\},\ a_{p-1} = l(q+1)+(p-1-l)q.$$

$I_0, \dots, I_{p-1}$ give a partition of $\{0, \dots, N\}$. Again there is an implicit dependence on N for each term defined above, and we would indicate this dependence explicitly when necessary. Then

$$a_t = tq + \min\{t, l\} = \begin{cases} t(q+1) & \text{when } j \in I_t,\ 0 \le t \le l, \\ l(q+1)+(t-l)q & \text{when } j \in I_t,\ l+1 \le t \le p-1, \end{cases}$$

$$\tau(j) = (j-a_t)p + t = \begin{cases} [j - t(q+1)]p + t & \text{when } j \in I_t,\ 0 \le t \le l, \\ [j - l(q+1) - (t-l)q]p + t & \text{when } j \in I_t,\ l+1 \le t \le p-1, \end{cases}$$

and if $\{j(N)\}_{N=1}^{\infty}$ is a sequence satisfying $j(N) \in I_t(N)$ for all $N \ge 1$, then

$$\left|\tau_N(j(N)) - pj(N) + t(N+1)\right| \le 2p^2,$$

and therefore

$$\frac{\tau_N(j(N))}{N+1} - \left(p\frac{j(N)}{N+1} - t\right) = O(N^{-1}). \tag{4.3}$$

Lemma 4.2. *With the assignment of the values of ξ_i given in table (4.2),*

$$\log|\det W_{m,N}(\xi)| = m\binom{N+m}{m+1}\log(\kappa r) + \frac{\beta_m}{p}N^{m+1} + o(N^{m+1}),$$

where $\beta_m = \frac{1}{(m-1)!}\int_0^1 x^m \log[2\sin(\pi x)]\ dx$, which is finite for each $m \ge 1$ by comparison test of *improper integrals.*

Proof. By Lemma 3.6,

$$\begin{aligned}
\log|\det W_{m,N}(\xi)| &= \log\Big[\prod_{i=1}^{m}\prod_{0\le j<k\le N}|\xi_{i,j}-\xi_{i,k}|^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}}\Big]\\
&=\sum_{i=1}^{m}\sum_{0\le j<k\le N}\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}\log\Big|\frac{\xi_{i,j}}{\kappa r}-\frac{\xi_{i,k}}{\kappa r}\Big|\\
&\quad+\sum_{i=1}^{m}\sum_{0\le j<k\le N}\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}\log(\kappa r)\\
&=\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log\Big|\frac{\xi_{i,\tau(j)}}{\kappa r}-\frac{\xi_{i,\tau(k)}}{\kappa r}\Big|\\
&\quad+m\binom{N+m}{m+1}\log(\kappa r)\\
&=\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log|e^{2\pi\sqrt{-1}\frac{j}{N+1}}-e^{2\pi\sqrt{-1}\frac{k}{N+1}}|\\
&\quad+m\binom{N+m}{m+1}\log(\kappa r)
\end{aligned}$$

where the second part of the third equality is due to (3.10). Now we are going to show that the first term after the last "=" can be approximated by a double integral.

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log|e^{2\pi\sqrt{-1}\frac{j}{N+1}}-e^{2\pi\sqrt{-1}\frac{k}{N+1}}|\\
=&\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\Big[\frac{(\tau(j))^{i-1}}{(i-1)!}+o((\tau(j))^{i-1})\Big]\Big[\frac{(N-\tau(k))^{m-i}}{(m-i)!}+o((N-\tau(k))^{m-i})\Big]\log|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}|.
\end{aligned}\tag{4.4}$$

For all $1\le i\le m, 0\le u,v\le p-1$, denote

$$L_{u,v,N}=\{(j,k)\in I_u\times I_v:\ \tau(j)<\tau(k)\},$$

$$T_{u,v}(N)=\bigcup_{(j,k)\in L_{u,v,N}}\Big[\frac{j}{N+1},\frac{j+1}{N+1}\Big]\times\Big[\frac{k}{N+1},\frac{k+1}{N+1}\Big],$$

$$\mathring{L}_{u,v,N}=\{(j,k)\in L_{u,v,N}:\ j-k\ne\pm N \text{ and } j-k\ne\pm1\}\subset L_{u,v,N},$$

$$\mathring{T}_{u,v}(N)=\bigcup_{(j,k)\in\mathring{L}_{u,v,N}}\Big[\frac{j}{N+1},\frac{j+1}{N+1}\Big]\times\Big[\frac{k}{N+1},\frac{k+1}{N+1}\Big]\subset T_{u,v}(N),$$

and a function defined over $\{(x,y)\in(0,1)\times(0,1):\ x\neq y\}$:

$$g^i_{u,v}(x,y)=(px-u)^{i-1}[1-(py-v)]^{m-i}\log|1-e^{2\pi\sqrt{-1}(x-y)}|.$$

Then

$$\left|L_{u,v,N}\smallsetminus \mathring{L}_{u,v,N}\right|\leq 2N+2, \tag{4.5}$$

$$\mathrm{Vol}_{\mathbb{R}^2}(T_{u,v}(N)\smallsetminus \mathring{T}_{u,v}(N))\leq O(N^{-1}), \tag{4.6}$$

$$\frac{1}{N+1}\leq\left|\frac{j-k}{N+1}\right|\leq\frac{N}{N+1}\ \text{ for } (j,k)\in L_{u,v,N}, \tag{4.7}$$

$$\frac{1}{N+1}\leq\left|x-y\right|\leq\frac{N}{N+1}\ \text{ for } (x,y)\in \mathring{T}_{u,v}(N), \tag{4.8}$$

$$|g^i_{u,v}(x,y)|\leq O(\log N)\ \text{ if } \frac{1}{N+1}\leq|x-y|\leq\frac{N}{N+1}, \tag{4.9}$$

$$\|\nabla g^i_{u,v}(x,y)\|\leq O(N^{\frac{1}{2}})\ \text{ if } \frac{1}{\sqrt{N+1}}\leq|x-y|\leq 1-\frac{1}{\sqrt{N+1}}. \tag{4.10}$$

From (4.3), we have

$$\begin{aligned}
&\sum_{0\leq\tau(j)<\tau(k)\leq N}(\tau(j))^{i-1}(N-\tau(k))^{m-i}\log|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}|\\
=&(N+1)^{m-1}\\
&\times\sum_{0\leq u,v\leq p-1}\sum_{(j,k)\in L_{u,v,N}}[p\frac{j}{N+1}-u+O(N^{-1})]^{i-1}[1-(p\frac{k}{N+1}-v)+O(N^{-1})]^{m-i}\log|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}|.
\end{aligned} \tag{4.11}$$

For all $0 \le u, v \le p-1$, by (4.5), (4.7) and (4.9), we get

$$\begin{aligned}
&\sum_{(j,k)\in L_{u,v,N}} (p\frac{j}{N+1}-u)^{i-1}[1-(p\frac{k}{N+1}-v)]^{m-i}\log|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}| \\
=&\sum_{(j,k)\in L_{u,v,N}} g^i_{u,v}(\frac{j}{N+1},\frac{k}{N+1}) \\
=&\sum_{(j,k)\in \mathring{L}_{u,v,N}} g^i_{u,v}(\frac{j}{N+1},\frac{k}{N+1}) + O(N\log N).
\end{aligned} \tag{4.12}$$

Moreover,

$$\begin{aligned}
&\Big|(N+1)^{-2}\sum_{(j,k)\in \mathring{L}_{u,v,N}} g^i_{u,v}(\frac{j}{N+1},\frac{k}{N+1}) - \iint_{\mathring{T}_{u,v}(N)} g^i_{u,v}(x,y)\ dxdy\Big| \\
\le &\sum_{(j,k)\in \mathring{L}_{u,v,N}} \iint_{[\frac{j}{N+1},\frac{j+1}{N+1}]\times[\frac{k}{N+1},\frac{k+1}{N+1}]} \Big|g^i_{u,v}(x,y)-g^i_{u,v}(\frac{j}{N+1},\frac{k}{N+1})\Big|\ dxdy \\
= &\sum_{(j,k)\in \mathring{L}_{u,v,N}:\frac{1}{\sqrt{N+1}}\le|\frac{j-k}{N+1}|\le 1-\frac{1}{\sqrt{N+1}}} \iint_{[\frac{j}{N+1},\frac{j+1}{N+1}]\times[\frac{k}{N+1},\frac{k+1}{N+1}]} \Big|g^i_{u,v}(x,y)-g^i_{u,v}(\frac{j}{N+1},\frac{k}{N+1})\Big|\ dxdy \\
&+\sum_{(j,k)\in \mathring{L}_{u,v,N}:|\frac{j-k}{N+1}|<\frac{1}{\sqrt{N+1}} \text{ or } |\frac{j-k}{N+1}|>1-\frac{1}{\sqrt{N+1}}} \iint_{[\frac{j}{N+1},\frac{j+1}{N+1}]\times[\frac{k}{N+1},\frac{k+1}{N+1}]} \Big|g^i_{u,v}(x,y)-g^i_{u,v}(\frac{j}{N+1},\frac{k}{N+1})\Big|\ dxdy.
\end{aligned} \tag{4.13}$$

Since

$$\sharp\left\{(j,k)\in \mathring{L}_{u,v,N}:\frac{1}{\sqrt{N+1}}\le|\frac{j-k}{N+1}|\le 1-\frac{1}{\sqrt{N+1}}\right\} \le \left|\mathring{L}_{u,v,N}\right| = O(N^2),$$

$$\sharp\left\{(j,k)\in \mathring{L}_{u,v,N}:|\frac{j-k}{N+1}|<\frac{1}{\sqrt{N+1}} \text{ or } |\frac{j-k}{N+1}|>1-\frac{1}{\sqrt{N+1}}\right\} \le O(N^{\frac{3}{2}}),$$

$$\begin{aligned}
(4.10)\ \Rightarrow &\sum_{(j,k)\in \mathring{L}_{u,v,N}:\frac{1}{\sqrt{N+1}}\le|\frac{j-k}{N+1}|\le 1-\frac{1}{\sqrt{N+1}}} \iint_{[\frac{j}{N+1},\frac{j+1}{N+1}]\times[\frac{k}{N+1},\frac{k+1}{N+1}]} \Big|g^i_{u,v}(x,y)-g^i_{u,v}(\frac{j}{N+1},\frac{k}{N+1})\Big|\ dxdy \\
&\le O(N^2)\times(N+1)^{-2}\times\frac{\sqrt{2}}{N+1}\times\sup_{\frac{1}{\sqrt{N+1}}\le|x-y|\le 1-\frac{1}{\sqrt{N+1}}}\|\nabla g^i_{u,v}(x,y)\| \\
&\le O(N^{-\frac{1}{2}}),
\end{aligned} \tag{4.14}$$

and by (4.8), (4.9),

$$\begin{aligned}
&\sum_{(j,k)\in\mathring{L}_{u,v,N}:|\frac{j-k}{N+1}|<\frac{1}{\sqrt{N+1}} \text{ or } |\frac{j-k}{N+1}|>1-\frac{1}{\sqrt{N+1}}} \iint_{[\frac{j}{N+1},\frac{j+1}{N+1}]\times[\frac{k}{N+1},\frac{k+1}{N+1}]} \left|g^i_{u,v}(x,y)-g^i_{u,v}(\frac{j}{N+1},\frac{k}{N+1})\right| \, dxdy \\
\le & O(N^{\frac{3}{2}})\times(N+1)^{-2}\times O(\log N) \\
= & O(N^{-\frac{1}{2}}\log N).
\end{aligned} \tag{4.15}$$

Denote $T_{u,v}=\{(x,y)\in\mathbb{R}^2:\ 0\le x-\frac{u}{p}\le y-\frac{v}{p}\le\frac{1}{p}\}$. Since $g^i_{u,v}$ is L^1_{loc}, the measure $g^i_{u,v}(x,y)\ dxdy$ is absolutely continuous with respect to the Lebesgue measure. Thus by lemma 4.3 below, we have

$$\iint_{\mathring{T}_{u,v}(N)} g^i_{u,v}(x,y)\ dxdy - \iint_{T_{u,v}} g^i_{u,v}(x,y)\ dxdy = o(1) \text{ as } N\to\infty. \tag{4.16}$$

$$\begin{aligned}
(4.12)\sim(4.16)\ \Rightarrow\ &\sum_{(j,k)\in L_{u,v,N}} (p\frac{j}{N+1}-u)^{i-1}[1-(p\frac{k}{N+1}-v)]^{m-i}\log|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}| \\
=&(N+1)^2\iint_{T_{u,v}} g^i_{u,v}(x,y)\ dxdy+o(N^2).
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
(4.17)+(4.11)\ \Rightarrow\ &\sum_{0\le\tau(j)<\tau(k)\le N} (\tau(j))^{i-1}(N-\tau(k))^{m-i}\log|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}| \\
=&(N+1)^{m+1}\sum_{0\le u,v\le p-1}\iint_{T_{u,v}} g^i_{u,v}(x,y)\ dxdy+o(N^{m+1}),
\end{aligned} \tag{4.18}$$

$$
\begin{aligned}
(4.18)+(4.4)\ \Rightarrow\ &\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log|e^{2\pi\sqrt{-1}\frac{j}{N+1}}-e^{2\pi\sqrt{-1}\frac{k}{N+1}}|\\
&=\sum_{i=1}^{m}\sum_{0\le u,v\le p-1}\iint_{T_{u,v}}\frac{g^i_{u,v}(x,y)}{(i-1)!(m-i)!}\,dxdy+o(N^{m+1})\\
&=\sum_{i=1}^{m}\sum_{0\le u,v\le p-1}\iint_{T_{u,v}}\frac{[p(x-\frac{u}{p})]^{i-1}}{(i-1)!}\frac{[1-p(y-\frac{v}{p})]^{m-i}}{(m-i)!}\log|1-e^{2\pi\sqrt{-1}(x-y)}|\,dxdy+o(N^{m+1})\\
&=\sum_{i=1}^{m}\sum_{0\le u,v\le p-1}\iint_{T_{0,0}}\frac{(px)^{i-1}}{(i-1)!}\frac{(1-py)^{m-i}}{(m-i)!}\log|1-e^{2\pi\sqrt{-1}(x-y+\frac{u}{p}-\frac{v}{p})}|\,dxdy+o(N^{m+1})\\
&=\sum_{i=1}^{m}\sum_{0\le u\le p-1}\iint_{T_{0,0}}\frac{(px)^{i-1}}{(i-1)!}\frac{(1-py)^{m-i}}{(m-i)!}\log\Big[\prod_{v=0}^{p-1}|e^{2\pi\sqrt{-1}\frac{v}{p}}-e^{2\pi\sqrt{-1}(x-y+\frac{u}{p})}|\Big]\,dxdy+o(N^{m+1})\\
&=p\sum_{i=1}^{m}\iint_{T_{0,0}}\frac{(px)^{i-1}}{(i-1)!}\frac{(1-py)^{m-i}}{(m-i)!}\log|1-e^{2\pi\sqrt{-1}(px-py)}|\,dxdy+o(N^{m+1})\\
&=\frac{1}{p}\iint_T\sum_{i=1}^{m}\frac{x^{i-1}}{(i-1)!}\frac{(1-y)^{m-i}}{(m-i)!}\log|1-e^{2\pi\sqrt{-1}(x-y)}|\,dxdy+o(N^{m+1})\\
&=\frac{1}{p(m-1)!}\iint_T(1+x-y)^{m-1}\log|1-e^{2\pi\sqrt{-1}(x-y)}|\,dxdy+o(N^{m+1}),
\end{aligned}
$$

where $T=\{(x,y)\in\mathbb{R}^2:\ 0\le x\le y\le 1\}$. After change of variables: $\tilde{x}=x-y,\ \tilde{y}=y$, T is mapped to $\tilde{T}=\{(\tilde{x},\tilde{y})\in\mathbb{R}^2:\ -1\le\tilde{x}\le 0,\ -\tilde{x}\le\tilde{y}\le 1\}$. Then,

$$
\begin{aligned}
&\frac{1}{(m-1)!}\iint_T(1+x-y)^{m-1}\log|1-e^{2\pi\sqrt{-1}(x-y)}|\,dxdy\\
=&\frac{1}{(m-1)!}\iint_{\tilde{T}}(1+\tilde{x})^{m-1}\log|1-e^{2\pi\sqrt{-1}\tilde{x}}|\,d\tilde{x}d\tilde{y}\\
=&\frac{1}{(m-1)!}\int_{-1}^{0}(1+\tilde{x})^{m}\log|1-e^{2\pi\sqrt{-1}\tilde{x}}|\,d\tilde{x}\\
=&\frac{1}{(m-1)!}\int_0^1 x^m\log|1-e^{2\pi\sqrt{-1}x}|\,dx\\
=&\frac{1}{(m-1)!}\int_0^1 x^m\log[2\sin(\pi x)]\,dx\\
=&\beta_m,
\end{aligned}
$$

hence,

$$
\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log|e^{2\pi\sqrt{-1}\frac{j}{N+1}}-e^{2\pi\sqrt{-1}\frac{k}{N+1}}|=\frac{\beta_m}{p}N^{m+1}+o(N^{m+1}).
$$

Thus,

$$
\log|\det W_{m,N}(\xi)|=m\binom{N+m}{m+1}\log(\kappa r)+\frac{\beta_m}{p}N^{m+1}+o(N^{m+1}).
$$

$\square$

Lemma 4.3. $\lim_{N\to\infty} \mathrm{Vol}_{\mathbb{R}^2}(T_{u,v} \triangle \mathring{T}_{u,v}(N)) = 0$ *for any* $0 \le u, v \le p-1$, *where* $T_{u,v} \triangle \mathring{T}_{u,v}(N)$ *denotes the difference set of* $T_{u,v}$ *and* $\mathring{T}_{u,v}(N)$.

Proof. By (4.6), the statement in the lemma is equivalent to $\lim_{N\to\infty} \mathrm{Vol}_{\mathbb{R}^2}(T_{u,v} \triangle T_{u,v}(N)) = 0$, which can be derived from $\lim_{N\to\infty}(T_{u,v}(N) \smallsetminus \partial T_{u,v}) = \mathring{T}_{u,v}$: $T_{u,v} \triangle T_{u,v}(N) = (T_{u,v} \smallsetminus T_{u,v}(N)) \cup (T_{u,v}(N) \smallsetminus T_{u,v})$. $T_{u,v} \smallsetminus T_{u,v}(N) \subset [\mathring{T}_{u,v} \smallsetminus (T_{u,v}(N) \smallsetminus \partial T_{u,v})] \cup \partial T_{u,v}$. Hence

$$
\begin{aligned}
\mathrm{Vol}_{\mathbb{R}^2}(T_{u,v} \smallsetminus T_{u,v}(N)) &\le \mathrm{Vol}_{\mathbb{R}^2}(\mathring{T}_{u,v} \smallsetminus (T_{u,v}(N) \smallsetminus \partial T_{u,v})) + \mathrm{Vol}_{\mathbb{R}^2}(\partial T_{u,v}) \\
&= \iint_{\mathbb{R}^2} \mathbb{1}_{\mathring{T}_{u,v} \smallsetminus (T_{u,v}(N) \smallsetminus \partial T_{u,v})}\, dxdy \\
&\le \iint_{\mathbb{R}^2} |\mathbb{1}_{\mathring{T}_{u,v}} - \mathbb{1}_{T_{u,v}(N) \smallsetminus \partial T_{u,v}}|\, dxdy,
\end{aligned}
$$

where the last line tends to 0 by Fatou's lemma. Similar proof works for $T_{u,v}(N) \smallsetminus T_{u,v}$. Therefore, it amounts to prove $\lim_{N\to\infty}(T_{u,v}(N) \smallsetminus \partial T_{u,v}) = \mathring{T}_{u,v}$.

First let's show $\limsup_{N\to\infty} T_{u,v}(N) \subset T_{u,v}$. For all $(x,y) \in \limsup_{N\to\infty} T_{u,v}(N)$, there exists a sequence $\{N_n\}_{n=1}^{\infty} \to \infty$ such that for any $n \ge 1$, there exists $(j(N_n), k(N_n)) \in I_u(N_n) \times I_v(N_n)$ with $\tau_{N_n}(j(N_n)) < \tau_{N_n}(k(N_n))$ and $(x,y) \in \left[\frac{j(N_n)}{N_n+1}, \frac{j(N_n)+1}{N_n+1}\right] \times \left[\frac{k(N_n)}{N_n+1}, \frac{k(N_n)}{N_n+1}\right]$. Then $\lim_{n\to\infty} \dfrac{j(N_n)}{N_n+1} = x$, $\lim_{n\to\infty} \dfrac{k(N_n)}{N_n+1} = y$. Since $0 \le \frac{\tau_{N_n}(j(N_n))}{N_n+1} < \frac{\tau_{N_n}(k(N_n))}{N_n+1} \le \frac{N_n}{N_n+1}$ and $(j(N_n), k(N_n)) \in I_u(N_n) \times I_v(N_n)$, (4.3) implies that $0 \le p \lim_{n\to\infty} \dfrac{j(N_n)}{N_n+1} - u \le p \lim_{n\to\infty} \dfrac{k(N_n)}{N_n+1} - v \le 1$. Hence $0 \le px - u \le py - v \le 1$ and $(x,y) \in T_{u,v}$.

Next we will prove $\mathring{T}_{u,v} \subset \liminf_{N\to\infty} T_{u,v}(N)$. For all $(x,y) \in \mathring{T}_{u,v}$, $0 < x - \frac{u}{p} < y - \frac{v}{p} < \frac{1}{p}$. Then there exists $0 < \epsilon_1, \epsilon_2, \eta_1, \eta_2 < \frac{1}{p}$ such that $x = \frac{u}{p} + \epsilon_1 = \frac{u+1}{p} - \eta_1$ and $y = \frac{v}{p} + \epsilon_2 = \frac{v+1}{p} - \eta_2$. For each $N > 0$, define $j(N) = \lfloor (N+1)x \rfloor$ and $k(N) = \lfloor (N+1)y \rfloor$. When N is large enough, $j(N) = \lfloor (N+1)(\frac{u}{p} + \epsilon_1) \rfloor = uq(N) + \lfloor u\frac{l(N)}{p} + \epsilon_1(N+1) \rfloor \ge uq(N) + \min\{u, l(N)\} = a_u$, while $j(N) = \lfloor (N+1)(\frac{u+1}{p} - \eta_1) \rfloor = (u+1)q(N) + \lfloor (u+1)\frac{l(N)}{p} - \eta_1(N+1) \rfloor \le (u+1)q(N) + \min\{u+1, l(N)\} - 1 = a_{u+1} - 1$ for $0 \le u < p-1$, which indicates that $j(N) \in I_u(N)$. And similarly, $k(N) \in I_v(N)$ for N large. Moreover, $\lim_{N\to\infty} \dfrac{\tau(j(N))}{N+1} = p \lim_{N\to\infty} \dfrac{j(N)}{N+1} - u = p \lim_{N\to\infty} \dfrac{\lfloor (N+1)x \rfloor}{N+1} - u = px - u$, similarly $\lim_{N\to\infty} \dfrac{\tau(k(N))}{N+1} = py - v$. And since $0 < px - u < py - v < 1$, for N large enough, $0 < \frac{\tau(j(N))}{N+1} < \frac{\tau(k(N))}{N+1} < 1 \Rightarrow 0 < \tau(j(N)) < \tau(k(N)) \le N$. Thus by the definition of $j(N)$ and $k(N)$, we have, for N large, $(x,y) \in [\frac{j(N)}{N+1}, \frac{j(N)+1}{N+1}] \times [\frac{k(N)}{N+1}, \frac{k(N)+1}{N+1}] \subset \bigcup_{(j,k) \in L_{u,v,N}} \left[\dfrac{j}{N+1}, \dfrac{j+1}{N+1}\right] \times \left[\dfrac{k}{N+1}, \dfrac{k+1}{N+1}\right] = T_{u,v}(N)$, which implies that $(x,y) \in \liminf_{N\to\infty} T_{u,v}(N)$.

In conclusion, we have

$$\mathring{T}_{u,v} \subset \liminf_{N\to\infty} T_{u,v}(N) \subset \limsup_{N\to\infty} T_{u,v}(N) \subset T_{u,v},$$
$$\Rightarrow \lim_{N\to\infty} (T_{u,v}(N) \setminus \partial T_{u,v}) = \mathring{T}_{u,v}.$$

□

Let $\zeta = (\zeta_J)^t_{J\in\Gamma_{m,N}} = (\tilde{s}_N(\xi_J))^t_{J\in\Gamma_{m,N}} = (\tilde{s}_N(\xi_{1,j_1},\dots,\xi_{m,j_m}))^t_{J\in\Gamma_{m,N}}$ be an $\binom{N+m}{m}$–dimensional mean zero complex Gaussian random vector. Denote its covariance matrix by Σ, then for all $J = (j_1,\dots,j_m), J' = (j'_1,\dots,j'_m) \in \Gamma_{m,N}$,

$$\begin{aligned}
\Sigma_{J,J'} &= \mathbb{E}_N(\zeta_J\bar{\zeta}_{J'}) = \mathbb{E}_N(\tilde{s}_N(\xi_J)\overline{\tilde{s}_N(\xi_{J'})}) \\
&= \sum_{K\in\Lambda_{m,N}} [\sqrt{\binom{N}{K}}\xi_J^K][\sqrt{\binom{N}{K}}\bar{\xi}_{J'}^K] \\
&= \sum_{K\in\Lambda_{m,N}} \binom{N}{K}(\xi_J\bar{\xi}_{J'})^K \\
&= (1+\xi_J\bar{\xi}_{J'})^N \\
&= (1+\xi_{1,j_1}\bar{\xi}_{1,j'_1}+\dots+\xi_{m,j_m}\bar{\xi}_{m,j'_m})^N,
\end{aligned}$$

where $\mathbb{E}_N$ denotes the expectation with respect to the probability measure γ_N.

Lemma 4.4. *With the assignment of ξ as in table (4.2),*

$$\log(\det\Sigma) = Q_{\kappa r,m}(N) + \frac{2\beta_m}{p}N^{m+1} + o(N^{m+1}).$$

Proof.

$$\Sigma = V_{m,N}(\xi)V^*_{m,N}(\xi),$$

where $V_{m,N}(\xi) = \left(\sqrt{\binom{N}{K}}\xi_J^K\right)_{J\in\Gamma_{m,N},\ K\in\Lambda_{m,N}}$ is an $\binom{N+m}{m}\times\binom{N+m}{m}$ matrix. Thus

$$\det\Sigma = |\det V_{m,N}(\xi)|^2 = \prod_{K\in\Lambda_{m,N}}\binom{N}{K}|\det W_{m,N}(\xi)|^2$$

By Lemma 4.2,

$$\begin{aligned}\log(\det\Sigma) &= \sum_{K\in\Lambda_{m,N}}\log\binom{N}{K} + 2\log|\det W_{m,N}(\xi)|\\ &= \sum_{K\in\Lambda_{m,N}}\log\binom{N}{K} + 2m\binom{N+m}{m+1}\log(\kappa r) + \frac{2\beta_m}{p}N^{m+1} + o(N^{m+1})\\ &= \sum_{K\in\Lambda_{m,N}}\log\binom{N}{K} + 2\sum_{K\in\Lambda_{m,N}}|K|\log(\kappa r) + \frac{2\beta_m}{p}N^{m+1} + o(N^{m+1})\\ &= Q_{\kappa r,m}(N) + \frac{2\beta_m}{p}N^{m+1} + o(N^{m+1}).\end{aligned}$$

□

As $\log|\tilde{s}_N(z)|$ is plurisubharmonic in a neighbourhood of $(\bar{D}(0,r))^m$, we have

$$\begin{aligned}&\log\prod_{J\in\Gamma_{m,N}}|\zeta_J|\\ =&\sum_{J\in\Gamma_{m,N}}\log|\tilde{s}_N(\xi_J)|\\ \le&\sum_{J\in\Gamma_{m,N}}\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\prod_{i=1}^m P_r(\xi_{i,j_i},u_i)\ d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\ =&(N+1)^m\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\Big[\sum_{J\in\Gamma_{m,N}}\prod_{i=1}^m\frac{P_r(\xi_{i,j_i},u_i)}{N+1} - \int_H\prod_{i=1}^m P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)\ d_mx\Big]\\ &d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\ &+(N+1)^m\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\int_H\prod_{i=1}^m P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)\ d_mx\ d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\ =&I+II,\end{aligned}\tag{4.19}$$

where $P_r(\xi,u)=\frac{r^2-|\xi|^2}{|u-\xi|^2}$ is the Poisson kernel of $D(0,r)$, $d\sigma_r$ is the Haar measure on $\partial D(0,r)$, d_mx is the Lebesgue measure on $\mathbb{R}^m$, and

$$H=\bigcup_{0\le t_1,\ldots,t_m\le p-1}H_{t_1,\ldots,t_m}:=\bigcup_{0\le t_1,\ldots,t_m\le p-1}\{x=(x_1,\ldots,x_m)\in\mathbb{R}^m:\ 0\le x_1-\frac{t_1}{p}\le\cdots\le x_m-\frac{t_m}{p}\le\frac{1}{p}\}.$$

$$\begin{aligned}I\le&(N+1)^m\max_{u\in(\partial D(0,r))^m}\Big|\sum_{J\in\Gamma_{m,N}}\prod_{i=1}^m\frac{P_r(\xi_{i,j_i},u_i)}{N+1}-\int_H\prod_{i=1}^m P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)\ d_mx\Big|\\ &\times\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)}\big|\log|\tilde{s}_N(u)|\big|\ d\sigma_r(u_1)\cdots d\sigma_r(u_m).\end{aligned}\tag{4.20}$$

First let's estimate $\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)}\big|\log|\tilde{s}_N(u)|\big|\ d\sigma_r(u_1)\cdots d\sigma_r(u_m)$.

Lemma 4.5. $\gamma_N\left(\sup_{u\in(\partial D(0,r))^m}|\tilde{s}_N(u)|<1\right)\le e^{-Q_{r,m}(N)}$.

Proof.

$$\tilde{s}_N(u)=\sum_{K\in\Lambda_{m,N}}c_K\sqrt{\binom{N}{K}}u^K$$
$$\Rightarrow\frac{\partial^K}{\partial u^K}\tilde{s}_N(0)=K!\sqrt{\binom{N}{K}}c_K,$$

where $\frac{\partial^K}{\partial u^K}$ refers to $\frac{\partial^{k_1}}{\partial u_1^{k_1}}\cdots\frac{\partial^{k_m}}{\partial u_1^{k_m}}$ and $K!=k_1!\cdots k_m!$.

By Cauchy's integral formula,

$$\frac{\partial^K}{\partial u^K}\tilde{s}_N(0)=\frac{K!}{(2\pi\sqrt{-1})^m}\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)}\frac{\tilde{s}_N(u)}{\prod_{i=1}^m u_i^{k_i+1}}\,du_1\cdots du_m,$$

$$\Rightarrow c_K=\binom{N}{K}^{-\frac{1}{2}}\frac{1}{(2\pi\sqrt{-1})^m}\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)}\frac{\tilde{s}_N(u)}{\prod_{i=1}^m u_i^{k_i+1}}\,du_1\cdots du_m,$$
$$\Rightarrow|c_K|\le\frac{\sup_{u\in(\partial D(0,r))^m}|\tilde{s}_N(u)|}{\sqrt{\binom{N}{K}}r^{|K|}},\qquad\text{for all }K\in\Lambda_{m,N}.$$

Therefore, $\sup_{u\in(\partial D(0,r))^m}|\tilde{s}_N(u)|<1$ would imply that for all $K\in\Lambda_{m,N}$,

$$|c_K|\le\left[\binom{N}{K}r^{2|K|}\right]^{-\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}\gamma_N\left(\sup_{u\in(\partial D(0,r))^m}|\tilde{s}_N(u)|<1\right)&\le\prod_{K\in\Lambda_{m,N}}\gamma_N\left(|c_K|\le\left[\binom{N}{K}r^{2|K|}\right]^{-\frac{1}{2}}\right)\\&\le\prod_{K\in\Lambda_{m,N}}\left[\binom{N}{K}r^{2|K|}\right]^{-1}\\&=e^{-Q_{r,m}(N)}.\end{aligned}$$

□

The next lemma follows directly from the first part of Theorem 3.1 in [7]. But here we provide a self-contained proof without using the language of sections and metrics.

Lemma 4.6. *Given $U \subset \mathbb{C}^m$ open and bounded with $\sup_{z \in \bar{U}} \|z\| = R > 0$, then for all $\eta > 0$,*

$$\gamma_N\{\sup_{z\in\bar{U}} |\tilde{s}_N(z)| > (1+R^2)^{\frac{N}{2}} e^{\eta N}\} \le e^{-e^{\eta N}}, \ \textit{for } N \gg 1.$$

Proof. By Cauchy-Schwartz inequality,

$$\begin{aligned}
\sup_{z\in\bar{U}} |\tilde{s}_N(z)| &= \sup_{z\in\bar{U}} \Big| \sum_{K\in\Lambda_{m,N}} c_K \sqrt{\binom{N}{K}} z^K \Big| \\
&\le \|c\| \sup_{z\in\bar{U}} \Big[\sum_{K\in\Lambda_{m,N}} \binom{N}{K} |z|^{2K} \Big]^{\frac{1}{2}} \\
&= \|c\| \sup_{z\in\bar{U}} (1+\|z\|^2)^{\frac{N}{2}} \\
&= \|c\| (1+R^2)^{\frac{N}{2}},
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \gamma_N\{\sup_{z\in\bar{U}} |\tilde{s}_N(z)| > (1+R^2)^{\frac{N}{2}} e^{\eta N}\} \\
&\le \gamma_N\{\|c\| > e^{\eta N}\} \\
&= e^{-e^{2\eta N}} \sum_{k=0}^{\binom{N+m}{m}-1} \frac{e^{(2\eta N)k}}{k!},
\end{aligned}$$

hence,

$$\begin{aligned}
&\log \gamma_N\{\sup_{z\in\bar{U}} |\tilde{s}_N(z)| > (1+R^2)^{\frac{N}{2}} e^{\eta N}\} \\
&\le -e^{2\eta N} + \log\binom{N+m}{m} + (2\eta N)\Big[\binom{N+m}{m} - 1\Big] \\
&\le -e^{\eta N}, \ \text{for } N \gg 1.
\end{aligned}$$

□

Lemma 4.7. $\int_{\partial D(0,r)} \cdots \int_{\partial D(0,r)} \big|\log|\tilde{s}_N(u)|\big| \, d\sigma_r(u_1)\cdots d\sigma_r(u_m) \le \dfrac{CN}{\delta^m}$ *for some constant C outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r,m}(N)}$.*

Proof. Applying Lemma 4.6 to $U = (D(0,r))^m$, we have

$$\gamma_N\{\sup_{u\in(\partial D(0,r))^m} |\tilde{s}_N(u)| > (1+mr^2)^{\frac{N}{2}} e^{\eta N}\} \le \gamma_N\{\sup_{u\in(\bar{D}(0,r))^m} |\tilde{s}_N(u)| > (1+mr^2)^{\frac{N}{2}} e^{\eta N}\} \le e^{-e^{\eta N}}. \tag{4.21}$$

Therefore, taking $\eta = 1$, outside an event of probability at most e^{-e^N}, we have

$$\begin{aligned}
&\log^+|\tilde{s}_N(u)| \le \tfrac{1}{2}N\log(1+mr^2) + N \text{ on } (\partial D(0,r))^m,\\
\Rightarrow &\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)} \log^+|\tilde{s}_N(u)|\ d\sigma_r(u_1)\cdots d\sigma_r(u_m) \le \tfrac{1}{2}N\log(1+mr^2)+N.
\end{aligned}\tag{4.22}$$

Applying Lemma 4.5 to the distinguished boundary $(\partial D(0,\kappa r))^m$, we have: outside an event of probability at most $e^{-Q_{\kappa r,m}(N)}$, $\sup_{u\in(\partial D(0,\kappa r))^m}|\tilde{s}_N(u)| \ge 1$, i.e. there exists some $\eta \in (\partial D(0,\kappa r))^m$ such that $|\tilde{s}_N(\eta)| \ge 1$,

$$\begin{aligned}
0 \le \log|\tilde{s}_N(\eta)| \le &\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\prod_{i=1}^m P_r(\eta_i,u_i)\ d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\
= &\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)} \log^+|\tilde{s}_N(u)|\prod_{i=1}^m P_r(\eta_i,u_i)\ d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\
&-\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)} \log^-|\tilde{s}_N(u)|\prod_{i=1}^m P_r(\eta_i,u_i)\ d\sigma_r(u_1)\cdots d\sigma_r(u_m),
\end{aligned}\tag{4.23}$$

Since for all $1 \le i \le m$, $|\eta_i| = \kappa r = (1-\sqrt{\delta})r$ and $|u_i| = r$, $\frac{\sqrt{\delta}}{2} \le P_r(\eta_i,u_i) \le \frac{2}{\sqrt{\delta}}$, (4.23) implies that outside an event of probability at most $e^{-Q_{\kappa r,m}(N)}$,

$$\begin{aligned}
&\Big(\frac{\sqrt{\delta}}{2}\Big)^m\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)} \log^-|\tilde{s}_N(u)|\ d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\
\le&\Big(\frac{2}{\sqrt{\delta}}\Big)^m\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)} \log^+|\tilde{s}_N(u)|\ d\sigma_r(u_1)\cdots d\sigma_r(u_m).
\end{aligned}\tag{4.24}$$

Combining (4.22) and (4.24), we get: outside an event of probability at most $e^{-e^N}+e^{-Q_{\kappa r,m}(N)}$,

$$\begin{aligned}
&\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)} \big|\log|\tilde{s}_N(u)|\big|\ d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\
=&\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)} \log^+|\tilde{s}_N(u)|\ d\sigma_r(u_1)\cdots d\sigma_r(u_m) + \int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)} \log^-|\tilde{s}_N(u)|\ d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\
\le&\Big[1+\Big(\frac{4}{\delta}\Big)^m\Big]\int_{\partial D(0,r)}\cdots\int_{\partial D(0,r)} \log^+|\tilde{s}_N(u)|\ d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\
\le&\Big[1+\Big(\frac{4}{\delta}\Big)^m\Big]\big[\tfrac{1}{2}N\log(1+mr^2)+N\big] = \frac{CN}{\delta^m}.
\end{aligned}$$

□

Lemma 4.8. $$\max_{u\in(\partial D(0,r))^m}\Big|\sum_{J\in\Gamma_{m,N}}\prod_{i=1}^m\frac{P_r(\xi_{i,j_i},u_i)}{N+1} - \int_H\prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\ d_mx\Big| \le \frac{o(1)}{\delta^{\frac{1}{2}(m+1)}}.$$

Proof. For all $u \in (\partial D(0,r))^m$,

$$
\begin{aligned}
&\Big|\sum_{J\in\Gamma_{m,N}}\prod_{i=1}^{m}\frac{P_r(\xi_{i,j_i},u_i)}{N+1}-\int_H\prod_{i=1}^{m}P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)\ d_mx\Big|\\
=&\Big|\sum_{\tau(J)\in\Gamma_{m,N}}\prod_{i=1}^{m}\frac{P_r(\xi_{i,\tau(j_i)},u_i)}{N+1}-\int_H\prod_{i=1}^{m}P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)\ d_mx\Big|\\
\le&\sum_{0\le t_1,\dots,t_m\le p-1}\Big|\sum_{J\in I_{t_1}\times\cdots\times I_{t_m}:\ \tau(J)\in\Gamma_{m,N}}\prod_{i=1}^{m}\frac{P_r(z_{j_i},u_i)}{N+1}-\int_{H_{t_1,\dots,t_m}}\prod_{i=1}^{m}P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)\ d_mx\Big|\\
\le&\sum_{0\le t_1,\dots,t_m\le p-1}\Big|\sum_{J\in I_{t_1}\times\cdots\times I_{t_m}:\ \tau(J)\in\Gamma_{m,N}}\prod_{i=1}^{m}\frac{P_r(z_{j_i},u_i)}{N+1}-\int_{H_{t_1,\dots,t_m}(N)}\prod_{i=1}^{m}P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)\ d_mx\Big|\\
&+\sum_{0\le t_1,\dots,t_m\le p-1}\Big|\int_{H_{t_1,\dots,t_m}(N)}\prod_{i=1}^{m}P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)\ d_mx-\int_{H_{t_1,\dots,t_m}}\prod_{i=1}^{m}P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)\ d_mx\Big|,
\end{aligned}
\tag{4.25}
$$

where $H_{t_1,\dots,t_m}(N)=\bigcup_{J\in I_{t_1}\times\cdots\times I_{t_m}:\ \tau(J)\in\Gamma_{m,N}}\big[\frac{j_1}{N+1},\frac{j_1+1}{N+1}\big]\times\cdots\times\big[\frac{j_m}{N+1},\frac{j_m+1}{N+1}\big]$.

For all $0\le t_1,\dots,t_m\le p-1$,

$$
\begin{aligned}
&\Big|\sum_{J\in I_{t_1}\times\cdots\times I_{t_m}:\ \tau(J)\in\Gamma_{m,N}}\prod_{i=1}^{m}\frac{P_r(z_{j_i},u_i)}{N+1}-\int_{H_{t_1,\dots,t_m}(N)}\prod_{i=1}^{m}P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)\ d_mx\Big|\\
\le&\sum_{J\in I_{t_1}\times\cdots\times I_{t_m}:\ \tau(J)\in\Gamma_{m,N}}\int_{\big[\frac{j_1}{N+1},\frac{j_1+1}{N+1}\big]\times\cdots\times\big[\frac{j_m}{N+1},\frac{j_m+1}{N+1}\big]}\Big|\prod_{i=1}^{m}P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)-\prod_{i=1}^{m}P_r(\kappa re^{2\pi\sqrt{-1}\frac{j_i}{N+1}},u_i)\Big|\ d_mx\\
\le&\frac{(q+1)^m}{(N+1)^m}m\sup_{|\omega|=\kappa r,|u|=r}[P_r(\omega,u)]^{m-1}\sup_{|\omega|\le\kappa r,|u|=r}\Big|\frac{\partial P_r(\omega,u)}{\partial\omega}\Big|\frac{2\pi\kappa r}{N+1}\\
\le&\frac{C}{p^m\delta^{\frac{1}{2}(m+1)}(N+1)},
\end{aligned}
$$

so

$$
\begin{aligned}
&\sum_{0\le t_1,\dots,t_m\le p-1}\Big|\sum_{J\in I_{t_1}\times\cdots\times I_{t_m}:\ \tau(J)\in\Gamma_{m,N}}\prod_{i=1}^{m}\frac{P_r(z_{j_i},u_i)}{N+1}-\int_{H_{t_1,\dots,t_m}(N)}\prod_{i=1}^{m}P_r(\kappa re^{2\pi\sqrt{-1}x_i},u_i)\ d_mx\Big|\\
\le&\frac{C}{\delta^{\frac{1}{2}(m+1)}(N+1)}=\frac{o(1)}{\delta^{\frac{1}{2}(m+1)}}
\end{aligned}
\tag{4.26}
$$

To bound the second term in (4.25), we need the following statement, which can be proved in a similar way as Lemma 4.3:

$$\lim_{N\to\infty}\mathrm{Vol}_{\mathbb{R}^m}(H_{t_1,\dots,t_m}(N)\ \triangle\ H_{t_1,\dots,t_m})=0 \text{ for any } 0\le t_1,\dots,t_m\le p-1.$$

$0<x_1-\frac{t_1}{p}<\cdots<x_m-\frac{t_m}{p}<\frac{1}{p}$. Then for $1\le i\le m$ there exist $\varepsilon_i,\eta_i>0$ such that $x_i=\frac{t_i}{p}+\varepsilon_i=\frac{t_i+1}{p}-$

η_i. For each $N > 0$, define $j_i(N) = \lfloor (N+1)x_i \rfloor$ for $1 \le i \le m$. When N is large enough, $\forall\ 1 \le i \le m$, $j_i(N) = \lfloor (N+1)(\frac{t_i}{p}+\varepsilon_i) \rfloor = t_i q(N) + \lfloor t_i \frac{l(N)}{p} + \varepsilon_i(N+1) \rfloor \ge t_i q(N) + \min\{t_i, l(N)\} = a_{t_i}$, while $j_i(N) = \lfloor (N+1)(\frac{t_i+1}{p} - \eta_i) \rfloor = (t_i+1)q(N) + \lfloor (t_i+1)\frac{l(N)}{p} - \eta_i(N+1) \rfloor \le (t_i+1)q(N) + \min\{t_i+1, l(N)\} - 1 = a_{t_i+1} - 1$, which indicates that $J(N) = (j_1(N), \dots, j_m(N)) \in I_{t_1}(N) \times \cdots \times I_{t_m}(N)$. Moreover, $\lim_{N\to\infty} \dfrac{\tau(j_i(N))}{N+1} = p \lim_{N\to\infty} \dfrac{j_i(N)}{N+1} - t_i = p \lim_{N\to\infty} \dfrac{\lfloor (N+1)x_i \rfloor}{N+1} - t_i = px_i - t_i$ and since $0 < px_1 - t_1 < \cdots < px_m - t_m < 1$, for N large enough, $0 < \frac{\tau(j_1(N))}{N+1} < \cdots < \frac{\tau(j_m(N))}{N+1} < 1$. Therefore $0 < \tau(j_1(N)) < \cdots < \tau(j_m(N)) \le N$ and $\tau(J(N)) \in \Gamma_{m,N}$. Thus by the definition of $J(N)$, we have, for N large, $x \in \big[\frac{j_1(N)}{N+1}, \frac{j_1(N)+1}{N+1}\big] \times \cdots \times \big[\frac{j_m(N)}{N+1}, \frac{j_m(N)+1}{N+1}\big] \subset \bigcup_{J \in I_{t_1}(N)\times\cdots\times I_{t_m}(N) : \tau(J) \in \Gamma_{m,N}} \big[\dfrac{j_1}{N+1}, \dfrac{j_1+1}{N+1}\big] \times \cdots \times \big[\dfrac{j_m}{N+1}, \dfrac{j_m+1}{N+1}\big] = H_{t_1,\dots,t_m}(N)$, which implies that $x \in \liminf_{N\to\infty} H_{t_1,\dots,t_m}(N)$.

Hence,

$$\begin{aligned}
&\sum_{0 \le t_1,\dots,t_m \le p-1} \Big| \int_{H_{t_1,\dots,t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i)\ d_m x - \int_{H_{t_1,\dots,t_m}} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i)\ d_m x \Big| \\
\le &\sum_{0 \le t_1,\dots,t_m \le p-1} \mathrm{Vol}_{R^m}(H_{t_1,\dots,t_m}(N)\ \triangle\ H_{t_1,\dots,t_m}) \Big[\sup_{|\omega|=\kappa r, |u|=r} P_r(\omega, u) \Big]^m \\
\le &\sum_{0 \le t_1,\dots,t_m \le p-1} o(1) \Big(\frac{2}{\sqrt{\delta}} \Big)^m \\
= &\frac{o(1)}{\delta^{\frac{1}{2}m}}.
\end{aligned} \tag{4.27}$$

This $o(1)$ may depend on p.

By (4.25), (4.26) and (4.27), the lemma is proved. □

Combining (4.20), Lemma 4.7 and Lemma 4.8, we have: outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r,m}(N)}$,

$$I \le (N+1)^m \frac{o(1)}{\delta^{\frac{1}{2}(m+1)}} \frac{CN}{\delta^m} = \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}.$$

By changing the order of integration,

$$II = (N+1)^m \int_H \int_{\partial D(0,r)} \cdots \int_{\partial D(0,r)} \log|\tilde{s}_N(u)| \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i)\ d\sigma_r(u_1)\cdots d\sigma_r(u_m)\ d_m x.$$

If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$, $\log|\tilde{s}_N(u)|$ is harmonic in $u_i \in$ a neighbourhood of $\bar{D}(0,r)$ for each fixed $(u_1, \dots, \hat{u}_i, \dots, u_m)$ in $(\bar{D}(0,r))^{m-1}$. Applying the mean value theorem for harmonic functions, we get

$$
\begin{aligned}
II =& (N+1)^m \times \\
& \int_H \int_{\partial D(0,r)} \cdots \int_{\partial D(0,r)} \log|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, u_2, \ldots, u_m)| \prod_{i=2}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i)\ d\sigma_r(u_2) \cdots d\sigma_r(u_m)\ d_m x \\
=& \ldots \\
=& (N+1)^m \int_H \log|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \ldots, \kappa r e^{2\pi\sqrt{-1}x_m})|\ d_m x.
\end{aligned}
$$

Denote

$$
\Xi = \int_H \log|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \ldots, \kappa r e^{2\pi\sqrt{-1}x_m})|\ d_m x, \tag{4.28}
$$

which is a complex random variable. Thus we have proved:

Lemma 4.9. *If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$, then outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r,m}(N)}$,*

$$
\log \prod_{J \in \Gamma_{m,N}} |\zeta_J| \le \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + (N+1)^m \Xi.
$$

Replacing $\Gamma_{m,N} = \{J = (j_1, \ldots, j_m) \in [0,N]^m \cap \mathbb{Z}^m : \ 0 \le j_1 \le \cdots \le j_m \le N\}$ by $\Gamma_{m,N}^{(\varrho)} = \{J = (j_1, \ldots, j_m) \in [0,N]^m \cap \mathbb{Z}^m : \ 0 \le j_{\varrho(1)} \le \cdots \le j_{\varrho(m)} \le N\}$, where ϱ can be any element in S_m, the permutation group of m letters, then similar results hold and we have counterparts for Lemma 4.4 and Lemma 4.9, which we state without proof.

Lemma 4.10. *Denote the covariance matrix of the random vector $(\zeta_J^{(\varrho)} = \tilde{s}_N(\xi_J))^t_{J \in \Gamma^{(\varrho)}_{m,N}}$ by $\Sigma^{(\varrho)}$. Then* $\log(\det \Sigma^{(\varrho)}) = Q_{\kappa r,m}(N) + \frac{2\beta_m}{p} N^{m+1} + o(N^{m+1})$.

For all $\varrho \in S_m$, denote

$$
\begin{aligned}
H^{(\varrho)} &= \bigcup_{0 \le t_1, \ldots, t_m \le p-1} H^{(\varrho)}_{t_1, \ldots, t_m} \\
&:= \bigcup_{0 \le t_1, \ldots, t_m \le p-1} \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : \ 0 \le x_{\varrho(1)} - \frac{t_{\varrho(1)}}{p} \le \cdots \le x_{\varrho(m)} - \frac{t_{\varrho(m)}}{p} \le \frac{1}{p}\}
\end{aligned}
$$

and the random variable

$$
\Xi^{(\varrho)} = \int_{H^{(\varrho)}} \log|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \ldots, \kappa r e^{2\pi\sqrt{-1}x_m})|\ d_m x.
$$

Then

Lemma 4.11. *If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$, then outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r,m}(N)}$,*

$$\log \prod_{J \in \Gamma^{(\varrho)}_{m,N}} |\zeta^{(\varrho)}_J| \leq \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + (N+1)^m \Xi^{(\varrho)}.$$

The last ingredient we need to prove the upper bound is the following lemma:

Lemma 4.12 ([4] Lemma 4.6)**.** *Let $s,t>0$ and $N \in \mathbb{N}^+$ such that $\log(t^N/s) \geq N$, then*

$$\mathrm{Vol}_{\mathbb{R}^N}\{(r_1,\dots,r_N) \in \mathbb{R}^N :\ 0 \leq r_j \leq t \text{ and } \Pi_{j=1}^N r_j \leq s\} \leq \frac{s}{(N-1)!} \log^N(t^N/s).$$

Proof of the upper bound in Theorem 1.1. If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$, by mean value property of pluriharmonic functions,

$$\begin{aligned}
\sum_{\varrho \in S_m} \Xi^{(\varrho)} &= \sum_{\varrho \in S_m} \int_{H^{(\varrho)}} \log|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1},\dots,\kappa r e^{2\pi\sqrt{-1}x_m})|\, d_m x \\
&= \int_{\bigcup_{\varrho \in S_m} H^{(\varrho)}} \log|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1},\dots,\kappa r e^{2\pi\sqrt{-1}x_m})|\, d_m x \\
&= \int_0^1 \cdots \int_0^1 \log|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1},\dots,\kappa r e^{2\pi\sqrt{-1}x_m})|\, dx_1 \cdots dx_m \\
&= \int_{\partial D(0,\kappa r)} \cdots \int_{\partial D(0,\kappa r)} \log|\tilde{s}_N(\omega_1,\dots,\omega_m)|\, d\sigma_{\kappa r}(\omega_1)\cdots d\sigma_{\kappa r}(\omega_m) \\
&= \log|\tilde{s}_N(0,\dots,0)| \\
&= \log|c_{(0,\dots,0)}|,
\end{aligned}$$

the second equality holds because for distinct $\varrho_1, \varrho_2 \in S_m$, $H^{(\varrho_1)} \cap H^{(\varrho_2)}$ is of m-dimensional Lebesgue measure zero. Then,

$$\begin{aligned}
P_{0,m}(r,N) =& \gamma_N\{0 \notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\} \\
=& \gamma_N\Big\{(\log|c_{(0,\dots,0)}| > 2m!\log N) \cap \Big(0 \notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\Big)\Big\} \\
&+ \gamma_N\Big\{(\log|c_{(0,\dots,0)}| \leq 2m!\log N) \cap \Big(0 \notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\Big)\Big\} \\
\leq& \gamma_N(|c_{(0,\dots,0)}| > N^{2m!}) + \gamma_N\Big\{\Big(\sum_{\varrho \in S_m} \Xi^{(\varrho)} \leq 2m!\log N\Big) \cap \Big(0 \notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\Big)\Big\} \\
\leq& e^{-N^{4m!}} + \gamma_N\Big\{\bigcup_{\varrho \in S_m} (\Xi^{(\varrho)} \leq 2\log N) \cap \Big(0 \notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\Big)\Big\} \\
\leq& e^{-N^{4m!}} + \sum_{\varrho \in S_m} \gamma_N\Big\{(\Xi^{(\varrho)} \leq 2\log N) \cap \Big(0 \notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\Big)\Big\}.
\end{aligned}$$

Lemma 4.9 implies

$$
\begin{aligned}
&\gamma_N\{(\Xi \le 2\log N) \cap \Big(0 \notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\Big)\} \\
\le& e^{-e^N} + e^{-Q_{\kappa r,m}(N)} + \gamma_N\{\log \prod_{J\in\Gamma_{m,N}} |\zeta_J| \le \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\} \\
=& e^{-e^N} + e^{-Q_{\kappa r,m}(N)} + \gamma_N\Big\{ \prod_{J\in\Gamma_{m,N}} |\zeta_J| \le \exp\{ \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\}\Big\}.
\end{aligned}
$$

Denote

$$
\mathcal{E}_{m,N} = \left\{ \zeta = (\zeta_J)_{J\in\Gamma_{m,N}} \in \mathbb{C}^{\binom{N+m}{m}} : \prod_{J\in\Gamma_{m,N}} |\zeta_J| \le \exp\Big\{ \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\Big\} \right\},
$$

and

$$
\mathcal{F}_{m,N} = \Big\{ \zeta = (\zeta_J)_{J\in\Gamma_{m,N}} \in \mathcal{E}_{m,N} : |\zeta_J| \le (2+2mr^2)^{\frac{N}{2}}, \ \forall \ J \in \Gamma_{m,N} \Big\} \subset \mathcal{E}_{m,N},
$$

both of which can be treated as subsets in $\mathbb{C}^{\binom{N+m}{m}}$ and events in the probability space $\big(H^0(\mathbb{CP}^m, \mathcal{O}(N)), \gamma_N\big)$. Thus,

$$
\begin{aligned}
\gamma_N\Big\{(\Xi \le 2\log N) \cap \Big(0 \notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\Big)\Big\} \le& e^{-e^N} + e^{-Q_{\kappa r,m}(N)} + \gamma_N(\mathcal{E}_{m,N}) \\
\le& e^{-e^N} + e^{-Q_{\kappa r,m}(N)} + \gamma_N(\mathcal{E}_{m,N} \setminus \mathcal{F}_{m,N}) + \gamma_N(\mathcal{F}_{m,N}).
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
\gamma_N(\mathcal{E}_{m,N} \setminus \mathcal{F}_{m,N}) \le& \gamma_N\Big\{|\zeta_J| > (2+2mr^2)^{\frac{N}{2}} \text{ for some } J \in \Gamma_{m,N}\Big\} \\
\le& \gamma_N\Big\{ \sup_{\omega\in(\partial D(0,\kappa r))^m} |\tilde{s}_N(\omega)| > (2+2mr^2)^{\frac{N}{2}} \Big\} \\
\le& \gamma_N\Big\{ \sup_{\omega\in(\bar{D}(0,r))^m} |\tilde{s}_N(\omega)| > (1+mr^2)^{\frac{N}{2}} 2^{\frac{N}{2}} \Big\} \\
\le& e^{-2^{\frac{N}{2}}},
\end{aligned}
\tag{4.30}
$$

where the last inequality is due to Lemma 4.6.

$$
\begin{aligned}
\gamma_N(\mathcal{F}_{m,N}) =& \frac{1}{\pi^{\binom{N+m}{m}} \det\Sigma} \int_{\mathcal{F}_{m,N}} e^{-\zeta^*\Sigma^{-1}\zeta} \ d_{2\binom{N+m}{m}}\zeta \\
\le& \exp\Big\{ -\Big[Q_{\kappa r,m}(N) + \frac{2\beta_m}{p} N^{m+1}\Big] + o(N^{m+1})\Big\} \pi^{-\binom{N+m}{m}} \mathrm{Vol}_{\mathbb{C}^{\binom{N+m}{m}}}(\mathcal{F}_{m,N})
\end{aligned}
$$

by Lemma 4.4. Change into polar coordinates and denote

$$
\begin{aligned}
&\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathcal{F}_{m,N})\\
=&\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}\Big\{(x_J)_{J\in\Gamma_{m,N}}\in\big[0,(2+2mr^2)^{\frac{N}{2}}\big]^{\binom{N+m}{m}}:\prod_{J\in\Gamma_{m,N}}x_J\le\exp\big\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\big\}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\gamma_N(\mathcal{F}_{m,N})\\
\le&2^{\binom{N+m}{m}}\exp\Big\{-\big[Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}\big]+o(N^{m+1})\Big\}\exp\big\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\big\}\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathcal{F}_{m,N})\\
=&2^{\binom{N+m}{m}}\exp\Big\{-\big[Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}\big]+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\Big\}\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathcal{F}_{m,N}).
\end{aligned}
$$

Since $\binom{N+m}{m}\frac{N}{2}\log(2+2mr^2)-\big[\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\big]>\binom{N+m}{m}$ for N large(up to now p,δ are constants), we can apply Lemma 4.12 and get:

$$
\begin{aligned}
&\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathcal{F}_{m,N})\\
\le&\frac{\exp\Big\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\Big\}}{[\binom{N+m}{m}-1]!}\Big\{\binom{N+m}{m}\frac{N}{2}\log(2+2mr^2)-\big[\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\big]\Big\}^{\binom{N+m}{m}}\\
\le&\frac{\exp\Big\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\Big\}}{2^{\binom{N+m}{m}}[\binom{N+m}{m}-1]!}\Big[N\binom{N+m}{m}\log(2+2mr^2)\Big]^{\binom{N+m}{m}},
\end{aligned}
$$

then,

$$
\begin{aligned}
\gamma_N(\mathcal{F}_{m,N})\le&\frac{\exp\Big\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N-[Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}]\Big\}}{[\binom{N+m}{m}-1]!}\\
&\times\Big[N\binom{N+m}{m}\log(2+2mr^2)\Big]^{\binom{N+m}{m}},
\end{aligned}
$$

so,

$$
\begin{aligned}
\log\gamma_N(\mathcal{F}_{m,N})\le&\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N-[Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}]\\
&+\binom{N+m}{m}\log\Big[N\binom{N+m}{m}\log(2+2mr^2)\Big]-\log\Big[\binom{N+m}{m}-1\Big]!\\
=&-Q_{\kappa r,m}(N)-\frac{2\beta_m}{p}N^{m+1}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}.
\end{aligned}
\tag{4.31}
$$

By Lemma 3.2, (4.29), (4.30) and (4.31),

$$\begin{aligned}
&\gamma_N\Big\{(\Xi \le 2\log N) \cap \Big(0 \notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\Big)\Big\}\\
\le& e^{-e^N} + e^{-Q_{\kappa r,m}(N)} + e^{-2^{\frac{N}{2}}} + \exp\Big\{-Q_{\kappa r,m}(N) - \frac{2\beta_m}{p}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\Big\}\\
\le& \exp\Big\{-\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}, \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\Big\}.
\end{aligned}$$

Similarly, for all $\varrho \in S_m$,

$$\begin{aligned}
&\gamma_N\Big\{(\Xi^{(\varrho)} \le 2\log N) \cap \Big(0 \notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\Big)\Big\}\\
\le& \exp\Big\{-\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}, \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\Big\},
\end{aligned}$$

thus,

$$\begin{aligned}
&P_{0,m}(r,N)\\
\le& e^{-N^{4m!}} + m!\exp\Big\{-\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}, \quad \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\Big\}\\
=& \exp\Big\{-\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}, \quad \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\Big\},\\
\Rightarrow& \log P_{0,m}(r,N) \le -\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}, \quad \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}},\\
\Rightarrow& \limsup_{N\to\infty}\frac{\log P_{0,m}(r,N)}{N^{m+1}} \le -\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}, \quad \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}.
\end{aligned}$$

Let $p \to \infty$, then

$$\limsup_{N\to\infty}\frac{\log P_{0,m}(r,N)}{N^{m+1}} \le -\Big[\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\Big].$$

Let $\delta \to 0+$, then $\kappa = 1 - \sqrt{\delta} \to 1$, so

$$\limsup_{N\to\infty}\frac{\log P_{0,m}(r,N)}{N^{m+1}} \le -\Big[\frac{2m\log r}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\Big].$$

Hence,

$$\log P_{0,m}(r,N) \le -\Big[\frac{2m\log r}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\Big]N^{m+1} + o(N^{m+1}).$$

Thus Theorem 1.1 is proved. $\square$

Chapter 5

Proof of Theorem 1.2

The proof of Theorem 1.2 is quite similar to that of Theorem 1.1. We only need to make some slight modifications in picking "determining exponents" and "sampling points".

5.1 Lower bound

Definition 5.1.

$$\Lambda_{m,N}(r) := \Big\{K \in \Lambda_{m,N} : \binom{N}{K} r^{2|K|} \geq 1\Big\} \subset \Lambda_{m,N},$$
$$R_{r,m}(N) := \sum_{K \in \Lambda_{m,N}(r)} \log\Big[\binom{N}{K} r^{2|K|}\Big].$$

Lemma 5.2. $\log P_{0,m}(r,N) \geq -R_{r,m}(N) + o(N^{m+1})$.

Proof. Consider the following event $\Omega_{r,m,N}$:

(i) $|c_{(0,\dots,0)}| \geq \sqrt{N}$,

(ii) $|c_K| \leq \dfrac{1}{2\sqrt{N}\sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}}$, $K \in \Lambda_{m,N}(r) \backslash \{(0,\dots,0)\}$,

(iii) $|c_K| \leq \dfrac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}}$, $K \in \Lambda_{m,N} \backslash \Lambda_{m,N}(r)$.

Then when $\Omega_{r,m,N}$ occurs, $\forall\ z \in (\bar{D}(0,r))^m$,

$$
\begin{aligned}
|\tilde{s}_N(z)| \geq \sqrt{N} &- \sum_{K\in\Lambda_{m,N}(r)\setminus\{(0,\dots,0)\}} \frac{\sqrt{\binom{N}{K}}r^{|K|}}{2\sqrt{N}\sqrt{\binom{N}{K}}r^{|K|}\binom{|K|+m-1}{m-1}} - \sum_{K\in\Lambda_{m,N}\setminus\Lambda_{m,N}(r)} \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}} \\
&= \sqrt{N} - \sum_{K\in\Lambda_{m,N}\setminus\{(0,\dots,0)\}} \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}} \\
&= \sqrt{N} - \sum_{k=1}^{N} \frac{1}{2\sqrt{N}} \\
&= \tfrac{1}{2}\sqrt{N} > 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
P_{0,m}(r,N) \geq &\gamma_N(\Omega_{r,m,N}) \\
&= \gamma_N\big(|c_{(0,\dots,0)}| \geq \sqrt{N}\big) \prod_{K\in\Lambda_{m,N}(r)\setminus\{(0,\dots,0)\}} \gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}}r^{|K|}\binom{|K|+m-1}{m-1}}\right) \\
&\quad\times \prod_{K\in\Lambda_{m,N}\setminus\Lambda_{m,N}(r)} \gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}}\right) \\
&\geq e^{-N} \prod_{K\in\Lambda_{m,N}(r)\setminus\{(0,\dots,0)\}} \frac{1}{8N\binom{N}{K}r^{2|K|}\binom{|K|+m-1}{m-1}^2} \prod_{K\in\Lambda_{m,N}\setminus\Lambda_{m,N}(r)} \frac{1}{8N\binom{|K|+m-1}{m-1}^2},
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \log P_{0,m}(r,N) \geq &-N - \sum_{K\in\Lambda_{m,N}(r)\setminus\{(0,\dots,0)\}} \log\Big[\binom{N}{K}r^{2|K|}\Big] - \sum_{K\in\Lambda_{m,N}(r)\setminus\{(0,\dots,0)\}} \log\Big[8N\binom{|K|+m-1}{m-1}^2\Big] \\
&= -\sum_{K\in\Lambda_{m,N}(r)\setminus\{(0,\dots,0)\}} \log\Big[\binom{N}{K}r^{2|K|}\Big] + o(N^{m+1}) \\
&= -R_{r,m}(N) + o(N^{m+1}).
\end{aligned}
$$

□

5.2 Upper bound

For some $\alpha \in (0,1]$, we can define the index sets $\Lambda_{m,\lfloor\alpha N\rfloor}$, $\Gamma_{m,\lfloor\alpha N\rfloor}$ and the $\binom{\lfloor\alpha N\rfloor+m}{m} \times \binom{\lfloor\alpha N\rfloor+m}{m}$ matrix

$$W_{m,\lfloor\alpha N\rfloor}(\xi) = (\xi_J^K)_{J\in\Gamma_{m,\lfloor\alpha N\rfloor},\ K\in\Lambda_{m,\lfloor\alpha N\rfloor}}.$$

We also assign the values of the variables $(\xi_{i,j})_{0\le i\le m,\ 0\le j\le\lfloor\alpha N\rfloor}$ by the points on $\partial D(0,\kappa r)$ in a way similar to §3 except that we replace N by $\lfloor\alpha N\rfloor$. Then we have the following lemma.

Lemma 5.3.

$$\log|\det W_{m,\lfloor\alpha N\rfloor}(\xi)| = m\binom{\lfloor\alpha N\rfloor+m}{m+1}\log(\kappa r) + \frac{\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1}).$$

$\zeta = (\zeta_J)^t_{J\in\Gamma_{m,\lfloor\alpha N\rfloor}} = (\tilde{s}_N(\xi_J))^t_{J\in\Gamma_{m,\lfloor\alpha N\rfloor}}$ is a dimension $\binom{\lfloor\alpha N\rfloor+m}{m}$ mean zero complex Gaussian random vector with covariance matrix

$$\Sigma = V_{m,N,\alpha}(\xi)V^*_{m,N,\alpha}(\xi),$$

where $V_{m,N,\alpha}(\xi) = \left(\sqrt{\binom{N}{K}}\xi_J^K\right)_{J\in\Gamma_{m,\lfloor\alpha N\rfloor},\ K\in\Lambda_{m,N}}$ is an $\binom{\lfloor\alpha N\rfloor+m}{m}\times\binom{N+m}{m}$ matrix.

Definition 5.4. $Q_{r,m,\alpha}(N) := \sum_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\log\left[\binom{N}{K}r^{2|K|}\right]$.

Lemma 5.5. $\log\det\Sigma \ge Q_{\kappa r,m,\alpha}(N) + \frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1})$.

Proof. By Cauchy-Binet identity,

$$\begin{aligned}
\det\Sigma &= \sum_{M:\ \binom{\lfloor\alpha N\rfloor+m}{m}\times\binom{\lfloor\alpha N\rfloor+m}{m}\text{ minor of }V_{m,N,\alpha}(\xi)}|\det M|^2\\
&\ge\left|\det\left(\sqrt{\binom{N}{K}}\xi_J^K\right)_{J\in\Gamma_{m,\lfloor\alpha N\rfloor},\ K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\right|^2\\
&= \prod_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\binom{N}{K}|\det W_{m,\lfloor\alpha N\rfloor}(\xi)|^2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\log\det\Sigma &\ge \sum_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\log\binom{N}{K} + 2m\binom{\lfloor\alpha N\rfloor+m}{m+1}\log(\kappa r) + \frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1})\\
&= \sum_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\log\left[\binom{N}{K}(\kappa r)^{2|K|}\right] + \frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1})\\
&= Q_{\kappa r,m,\alpha}(N) + \frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1}).
\end{aligned}$$

□

The following lemma is a counterpart of Lemma 4.9. The proof is similar.

Lemma 5.6. *If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$, then outside an event of probability at most*

$e^{-e^N} + e^{-R_{\kappa r,m}(N)}$,

$$\log \prod_{J \in \Gamma_{m,\lfloor \alpha N \rfloor}} |\zeta_J| \le \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + (\lfloor \alpha N \rfloor + 1)^m \Xi,$$

where the complex random variable Ξ is defined in (4.28).

By playing the same trick of permutation as in §3, we can get an upper bound estimate for $P_{0,m}(r, N)$:

$$P_{0,m}(r,N) \le e^{-N^{4m!}} + m! \Big\{ e^{-e^N} + e^{-R_{\kappa r,m}(N)} + e^{-2^{\frac{N}{2}}} + \exp\Big[- Q_{\kappa r,m,\alpha}(N) - \frac{2\beta_m}{p} (\lfloor \alpha N \rfloor)^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} \Big] \Big\}. \tag{5.1}$$

5.3 Punch line of the proof

In order to prove Theorem 1.2, it suffices to compute $R_{r,m}(N)$ and $Q_{r,m,\alpha}(N)$ asymptotically. We follow the same idea as that in Lemma 3.2.

The scaled lattice $\frac{1}{N}\Lambda_{m,N}(r)$ corresponds to the set

$$\{x = (x_1, \ldots, x_m) \in \Sigma_m : \ E_r(x) \ge 0\}$$

and $\frac{1}{N}\Lambda_{r,m,\alpha}(N)$ corresponds to the set

$$\{x = (x_1, \ldots, x_m) \in \mathbb{R}^{m+} : \ \sum_{i=1}^m x_i \le \alpha \le 1\}.$$

So we have

$$R_{r,m}(N) = \sum_{K \in \Lambda_{m,N}(r)} \log \Big[\binom{N}{K} r^{2|K|} \Big] = N^{m+1} \int_{x \in \Sigma_m : \ E_r(x) \ge 0} E_r(x) \ d_m x + o(N^{m+1}), \tag{5.2}$$

$$Q_{r,m,\alpha}(N) = \sum_{K \in \Lambda_{m,\lfloor \alpha N \rfloor}} \log \Big[\binom{N}{K} r^{2|K|} \Big] = N^{m+1} \int_{x \in \mathbb{R}^{m+} : \ \sum_{i=1}^m x_i \le \alpha} E_r(x) \ d_m x + o(N^{m+1}). \tag{5.3}$$

Moreover, if we go through the proof of Lemma 3.2, we find that the $o(N^{m+1})$ terms in (5.2) and (5.3) are uniform if $r \le c$ for some constant $c > 0$, which implies that when r is replaced by $\kappa r = (1 - \sqrt{\delta}) r$, the remainder won't depend on δ.

Proof of Theorem 1.2. The lower bound proof is already implied by Lemma 5.2 and (5.2). To prove the upper bound, by (5.1) and (5.3),

$$\begin{aligned}&\log P_{0,m}(r,N)\\ \le& -N^{m+1}\min\Big\{\int_{x\in\Sigma_m:\ E_{\kappa r}(x)\ge 0} E_{\kappa r}(x)\ d_m x,\ \int_{x\in\mathbb{R}^{m+}:\ \sum_{i=1}^m x_i\le\alpha} E_{\kappa r}(x)\ d_m x+\frac{2\beta_m\alpha^{m+1}}{p}\Big\}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}.\end{aligned}$$

Similar as in §3, we can get

$$\begin{aligned}\log P_{0,m}(r,N)&\le -N^{m+1}\min\Big\{\int_{x\in\Sigma_m:\ E_r(x)\ge 0} E_r(x)\ d_m x,\int_{x\in\mathbb{R}^{m+}:\ \sum_{i=1}^m x_i\le\alpha} E_r(x)\ d_m x\Big\}+o(N^{m+1})\\ &=-N^{m+1}\int_{x\in\mathbb{R}^{m+}:\ \sum_{i=1}^m x_i\le\alpha} E_r(x)\ d_m x+o(N^{m+1}).\end{aligned}$$

Now the question amounts to find a proper $\alpha_0=\alpha_0(r,m)\in(0,1]$ which maximizes $\int_{x\in\mathbb{R}^{m+}:\ \sum_{i=1}^m x_i\le\alpha} E_r(x)\,d_m x$. For this purpose we consider the function defined on $(0,1]$

$$\Upsilon(\alpha):=\int_{x\in\mathbb{R}^{m+}:\ \sum_{i=1}^m x_i\le\alpha} E_r(x)\ d_m x.$$

Then

$$\begin{aligned}\Upsilon(\alpha)=&2m\log r\int_{x\in\mathbb{R}^{m+}:\ \sum_{i=1}^m x_i\le\alpha} x_1\ d_m x-m\int_{x\in\mathbb{R}^{m+}:\ \sum_{i=1}^m x_i\le\alpha} x_1\log x_1\ d_m x\\ &-\int_{x\in\mathbb{R}^{m+}:\ \sum_{i=1}^m x_i\le\alpha}(1-\sum_{i=1}^m x_i)\log(1-\sum_{i=1}^m x_i)\ d_m x\\ =&2m\log r\frac{\alpha^{m+1}}{(m+1)!}-m\frac{\alpha^{m+1}}{(m+1)!}\Big[\log\alpha-\sum_{k=2}^{m+1}\frac{1}{k}\Big]-\frac{1}{(m-1)!}\int_0^\alpha(1-x)x^{m-1}\log(1-x)\ dx,\end{aligned}$$

$$\Upsilon'(\alpha)=\frac{\alpha^{m-1}}{(m-1)!}\Big\{\big(2\log r+\sum_{k=2}^m\frac{1}{k}\big)\alpha-\big[\alpha\log\alpha+(1-\alpha)\log(1-\alpha)\big]\Big\},$$

where we take $\sum_{k=2}^m\frac{1}{k}=0$ when $m=1$. So if $2\log r+\sum_{k=2}^m\frac{1}{k}\ge 0$, $\Upsilon'(\alpha)\ge 0$ over $(0,1]$,

$$\max_{(0,1]}\Upsilon=\Upsilon(1),\ \Rightarrow\alpha_0=1.$$

If $2\log r+\sum_{k=2}^{m}\frac{1}{k}<0$, let $\alpha_0\in(0,1)$ be the nonzero root of $\big(2\log r+\sum_{k=2}^{m}\frac{1}{k}\big)\alpha=\alpha\log\alpha+(1-\alpha)\log(1-\alpha)$,

$$
\begin{aligned}
\max_{(0,1]}\Upsilon=\Upsilon(\alpha_0)&=\int_{x\in\mathbb{R}^{m+}:\ \sum_{i=1}^{m}x_i\le\alpha_0}E_r(x)\ d_m x\\
\Big(&=\frac{1}{(m+1)!}\big[(1-\alpha_0^m)\log(1-\alpha_0)+\sum_{k=1}^{m}\frac{\alpha_0^k}{k}\big]\Big).
\end{aligned}
\tag{5.4}
$$

□

Remark 5.7. *The proofs of Theorem 1.1 and 1.2 also work for a general polydisk* $\prod_{i=1}^{m}D(0,r_i)$*. For example, if* $r=(r_1,\dots,r_m)\in[1,\infty)^m$*, the function* E_r *in Theorem 1.1 would be* $E_r(x)=2\sum_{i=1}^{m}x_i\log r_i-[\sum_{i=1}^{m}x_i\log x_i+(1-\sum_{i=1}^{m}x_i)\log(1-\sum_{i=1}^{m}x_i)]$ *and* $\int_{\Sigma_m}E_r(x)\ d_m x=\frac{2}{(m+1)!}\sum_{i=1}^{m}\log r_i+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}$.

Chapter 6

Hole probability of $SU(2)$ polynomials

Proof of Corollary 1.4. When $r \geq 1$, $\alpha_0 = 1$. The result follows from Theorem 1.1.

When $0 < r < 1$,

$$x \in \mathbb{R}^+ : \ E_r(x) = 2x \log r - \big[x \log x + (1-x) \log (1-x) \big] \geq 0 \Leftrightarrow 0 \leq x \leq \alpha_0.$$

By Theorem 1.2,

$$\log P_{0,1}(r, N) = -N^2 \int_0^{\alpha_0} E_r(x) \ dx + o(N^2),$$

where the value of the integral in the corollary is due to (5.4) and the fact that

$$2\alpha_0 \log r = \alpha_0 \log \alpha_0 + (1-\alpha_0) \log (1-\alpha_0).$$

□

Proof of Theorem 1.5. Since ∂U is a Jordan curve, by Carathéodory's theorem, ϕ can be extended to a homeomorphism $\bar{D}(0,1) \to \bar{U}$. We still use ϕ to denote the extended map. Thus, $\tilde{s}_N(z) = \sum_{k=0}^{N} c_k \sqrt{\binom{N}{k}} z^k$ is nonvanishing over $\bar{U}$ if and only if $t_N(\omega) := \sum_{k=0}^{N} c_k \sqrt{\binom{N}{k}} (\phi(\omega))^k$ is nonvanishing over $\bar{D}(0,1)$, where $t_N \in \mathcal{O}(D(0,1)) \cap \mathcal{C}(\bar{D}(0,1))$.

Since

$$\begin{bmatrix} t_N(0) \\ t'_N(0) \\ \vdots \\ t_N^{(N)}(0) \end{bmatrix} = A \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix},$$

where A is an $(N+1)\times(N+1)$ lower triangular matrix with diagonal entries $\left\{k!\sqrt{\binom{N}{k}}(\phi'(0))^k\right\}_{0\le k\le N}$, $\left(t_N(0) \ \dots \ t_N^{(N)}(0)\right)^t$ is Gaussian with covariance matrix AA^*.

$$\det(AA^*) = |\det A|^2 = \prod_{k=0}^{N}\left[k!^2\binom{N}{k}|\phi'(0)|^{2k}\right] \neq 0 \tag{6.1}$$

because ϕ is a biholomorphism.

We again define $\kappa = 1-\sqrt{\delta}$. Then if $\sup\limits_{\partial D(0,\kappa)}|t_N| < 1$, for $0 \le k \le N$,

$$|t_N^{(k)}(0)| = \Big|\frac{k!}{2\pi\sqrt{-1}}\int_{\partial D(0,\kappa)}\frac{t_N(u)}{u^{k+1}}\,du\Big| \le \frac{k!}{\kappa^k}.$$

Therefore,

$$\begin{aligned}
\gamma_N\big(\sup_{\partial D(0,\kappa)}|t_N| < 1\big) &\le \gamma_N\Big\{\big(t_N(0),\dots,t_N^{(N)}(0)\big) \in \prod_{k=0}^{N}\bar D\big(0,\frac{k!}{\kappa^k}\big)\Big\} \\
&= \frac{1}{\pi^{N+1}\det(AA^*)}\int_{\prod_{k=0}^{N}\bar D\left(0,\frac{k!}{\kappa^k}\right)}\exp\{-\eta^*(AA^*)^{-1}\eta\}\ d_{2(N+1)}\eta \\
&\le \frac{\pi^{N+1}\prod_{k=0}^{N}\left(\frac{k!}{\kappa^k}\right)^2}{\pi^{N+1}\det(AA^*)}.
\end{aligned}$$

By (6.1),

$$\begin{aligned}
\gamma_N\big(\sup_{\partial D(0,\kappa)}|t_N| < 1\big) &\le \frac{\prod_{k=0}^{N}\left(\frac{k!}{\kappa^k}\right)^2}{\prod_{k=0}^{N}\left[k!^2\binom{N}{k}|\phi'(0)|^{2k}\right]} \\
&= \Big\{\prod_{k=0}^{N}\Big[\binom{N}{k}(\kappa|\phi'(0)|)^{2k}\Big]\Big\}^{-1} \\
&= \exp\{-Q_{\kappa|\phi'(0)|,1}(N)\} \\
&= \exp\{-(\log|\phi'(0)| + \log\kappa + \tfrac{1}{2})N^2 + o(N^2)\},
\end{aligned}$$

where the last equality is due to Lemma 3.2.

Similar as Lemma 4.9, we can show that if $t_N|_{\bar D(0,1)} \neq 0$, then outside an event of probability at most $e^{-e^N} + \exp\{-Q_{\kappa|\phi'(0)|,1}(N)\} = \exp\{-(\log|\phi'(0)| + \log\kappa + \frac{1}{2})N^2 + o(N^2)\}$,

$$\log\prod_{j=0}^{N}|t_N(z_j)| \le \frac{o(N^2)}{\delta^2} + (N+1)\log|c_0|,$$

where $z_j = \kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}}$, $0 \le j \le N$.

$\big(t_N(z_0)\ \dots\ t_N(z_N)\big)^t$ is complex Gaussian with covariance matrix

$$\Sigma = \big(\mathbb{E}_N(t_N(z_j)\overline{t_N(z_j)})\big)_{0\le i,j\le N} = \big(\sum_{k=0}^{N}\binom{N}{k}(\phi(z_i))^k(\overline{\phi(z_j)})^k\big)_{0\le i,j\le N}$$
$$= \begin{bmatrix} \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_0) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_0))^N \\ \cdots & \cdots & \cdots & \cdots \\ \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_N) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_N))^N \end{bmatrix}\begin{bmatrix} \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_0) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_0))^N \\ \cdots & \cdots & \cdots & \cdots \\ \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_N) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_N))^N \end{bmatrix}^*$$

and

$$\det\Sigma = \prod_{k=0}^{N}\binom{N}{k}\prod_{0\le i<j\le N}|\phi(z_i)-\phi(z_j)|^2,$$

$$\Rightarrow \log\det\Sigma = \sum_{k=0}^{N}\log\binom{N}{k} + 2\sum_{0\le i<j\le N}\log|\phi(z_i)-\phi(z_j)|, \tag{6.2}$$

Next we will show that

$$2\sum_{0\le i<j\le N}\log|\phi(z_i)-\phi(z_j)| = N^2\int_{\partial D(0,\kappa)}\int_{\partial D(0,\kappa)}\log|\phi(u_1)-\phi(u_2)|\ d\sigma_\kappa(u_1)d\sigma_\kappa(u_2) + o_\delta(N^2), \tag{6.3}$$

where $o_\delta(N^2)$ denotes a lower order term depending on δ.

Since

$$2\sum_{0\le i<j\le N}\log|\phi(z_i)-\phi(z_j)| = 2(N+1)^2\sum_{0\le i<j\le N}\frac{1}{(N+1)^2}\log|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})|$$

and

$$\begin{aligned}
&\int_{\partial D(0,\kappa)}\int_{\partial D(0,\kappa)} \log|\phi(u_1)-\phi(u_2)|\, d\sigma_\kappa(u_1)d\sigma_\kappa(u_2)\\
=&\int_0^1\int_0^1 \log|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})|\, dxdy\\
=&2\iint_{0\le x\le y\le 1} \log|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})|\, dxdy,
\end{aligned}$$

it suffices to show that

$$\Big|\sum_{0\le i<j\le N}\frac{1}{(N+1)^2}\log|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})|-\iint_{0\le x\le y\le 1}\log|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})|\, dxdy\Big|$$
$$=o_\delta(1).$$

Since ϕ is a biholomorphism, we set

$$\inf_{\bar{D}(0,\kappa)}|\phi'|=a(\delta)>0.$$

And by Cauchy's inequality, we have

$$\sup_{\bar{D}(0,\kappa)}|\phi'|\le O(\delta^{-1}).$$

For each N, denote

$$\Delta(N)=\{(i,j)\in\mathbb{Z}^2:\ 0\le i<j\le N\},$$

the "far from diagonal" indices

$$FD(N)=\left\{(i,j)\in\Delta(N):\begin{array}{cc}\lfloor\sqrt{N+1}\rfloor+i\le j\le N-\lfloor\sqrt{N+1}\rfloor+i & \text{if } 0\le i\le\lfloor\sqrt{N+1}\rfloor\\ \lfloor\sqrt{N+1}\rfloor+i\le j\le N & \text{if } \lfloor\sqrt{N+1}\rfloor<i\le N-\lfloor\sqrt{N+1}\rfloor\\ j\in\varnothing & \text{if } i>N-\lfloor\sqrt{N+1}\rfloor\end{array}\right\},$$

$$\mathscr{F}\mathscr{D}(N)=\bigcup_{(i,j)\in FD(N)}[\frac{i}{N+1},\frac{i+1}{N+1}]\times[\frac{j}{N+1},\frac{j+1}{N+1}],$$

and the "near diagonal" indices:

$$D(N)=\Delta(N)\smallsetminus FD(N).$$

Then

$$|D(N)| = O(N^{\frac{3}{2}}),$$

and for $(i,j) \in FD(N)$,

$$\frac{i}{N+1} - \frac{j}{N+1} \geq (N+1)^{-\frac{1}{2}} \quad \mod 1.$$

So

$$\begin{aligned}
&\Big|\sum_{0\leq i<j\leq N} \frac{1}{(N+1)^2} \log|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})| - \iint_{0\leq x\leq y\leq 1} \log|\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})|\, dxdy\Big| \\
\leq &\sum_{(i,j)\in D(N)} \frac{1}{(N+1)^2}\Big|\log|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})|\Big| \\
&+ \sum_{(i,j)\in FD(N)} \int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}} \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \Big|\log|\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})| - \log|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})|\Big| dxdy \\
&+ \Big|\iint_{\mathscr{F}\mathscr{D}(N)} \log|\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})|\, dxdy - \iint_{0\leq x\leq y\leq 1} \log|\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})|\, dxdy\Big| \\
= &I + II + III.
\end{aligned}$$

$$\begin{aligned}
&\frac{a(\delta)}{N+1} \leq |\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})| \leq O(1) \quad \forall (i,j) \in D(N), \\
\Rightarrow &\Big|\log|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})|\Big| \leq |\log a(\delta)| + \log(N+1), \\
\Rightarrow &I \leq \frac{O(N^{\frac{3}{2}})}{N^2}[|\log a(\delta)| + \log(N+1)] = o_\delta(1).
\end{aligned}$$

Since

$$\sup_{x-y\geq (N+1)^{-\frac{1}{2}} \mod 1} \|\nabla \log|\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})|\| \leq \frac{O(\delta^{-1})}{a(\delta)(N+1)^{-\frac{1}{2}}} = \frac{O(N^{\frac{1}{2}})}{\delta a(\delta)},$$

$$\begin{aligned}
\Rightarrow II \leq &\frac{N^2}{(N+1)^2} \sup_{x-y\geq (N+1)^{-\frac{1}{2}} \mod 1} \|\nabla \log|\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})|\| O(N^{-1}) \\
\leq &\frac{O(N^{-\frac{1}{2}})}{\delta a(\delta)} = o_\delta(1).
\end{aligned}$$

By a similar argument as Lemma 4.3, we have

$$\lim_{N\to\infty} \mathrm{Vol}_{\mathbb{R}^2}\left(\mathscr{F}\mathscr{D}(N) \triangle \{(x,y)\in\mathbb{R}^2:\ 0\le x\le y\le 1\}\right) = 0.$$

Furthermore, (6.4) and (6.5) below indicate that the function $\log|\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})|$ is L^1 over $[0,1]^2$,

$$\Rightarrow III \le o_\delta(1).$$

Thus, we have proved (6.3).

For $u_1,\ u_2 \in D(0,1)$, define:

$$\psi(u_1,u_2) = \begin{cases} \frac{\phi(u_1)-\phi(u_2)}{u_1-u_2} & \text{if } u_1 \ne u_2, \\ \phi'(u_1) & \text{if } u_1 = u_2. \end{cases}$$

Then ψ is continuous and nonzero in $D(0,1)\times D(0,1)$. Moreover, by removable singularity theorem, ψ is holomorphic in u_1 as well as u_2. Therefore, $\log|\psi|$ is pluriharmonic in $D(0,1)\times D(0,1)$. By mean value equality,

$$\begin{aligned}
&\int_{\partial D(0,\kappa)}\int_{\partial D(0,\kappa)} \log|\phi(u_1)-\phi(u_2)|\ d\sigma_\kappa(u_1)d\sigma_\kappa(u_2)\\
=&\int_{\partial D(0,\kappa)}\int_{\partial D(0,\kappa)} \log|\psi(u_1,u_2)|\ d\sigma_\kappa(u_1)d\sigma_\kappa(u_2) + \int_{\partial D(0,\kappa)}\int_{\partial D(0,\kappa)} \log|u_1-u_2|\ d\sigma_\kappa(u_1)d\sigma_\kappa(u_2)\\
=&\log|\psi(0,0)| + \log\kappa + \int_{\partial D(0,1)}\int_{\partial D(0,1)} \log|u_1-u_2|\ d\sigma_1(u_1)d\sigma_1(u_2)\\
=&\log|\phi'(0)| + \log\kappa + \int_{\partial D(0,1)}\int_{\partial D(0,1)} \log|u_1-u_2|\ d\sigma_1(u_1)d\sigma_1(u_2),
\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
&\int_{\partial D(0,1)}\int_{\partial D(0,1)} \log|u_1-u_2|\ d\sigma_1(u_1)d\sigma_1(u_2)\\
=&\int_0^1\int_0^1 \log|e^{2\pi\sqrt{-1}x} - e^{2\pi\sqrt{-1}y}|\ dxdy\\
=&\int_0^1 \log|1-e^{2\pi\sqrt{-1}x}|\ dx\\
=&\int_{\partial D(0,1)} \log|1-z|\ d\sigma_1(z)\\
=&0,
\end{aligned} \tag{6.5}$$

where the last equality is due to Lebesgue's dominated convergence theorem.

(6.2)~(6.5) show that

$$\begin{aligned}\log\det\Sigma &= \sum_{k=0}^{N}\log\binom{N}{k} + (\log|\phi'(0)| + \log\kappa)N^2 + o_\delta(N^2)\\ &= (\log|\phi'(0)| + \log\kappa + \tfrac{1}{2})N^2 + o_\delta(N^2).\end{aligned}$$

The remaining part is similar to §3. □

Remark 6.1. *For $U = D(0,r)$, ϕ would be a rotation composed with a scaling by r. So $|\phi'(0)| = r$. Thus the upper bound in Theorem 1.5 is $-(\log r + \frac{1}{2})N^2 + o(N^2)$, which agrees with Corollary 1.4 in the case of $r \geq 1$.*

Chapter 7

Generalized hole probabilities of $SU(2)$ polynomials

If $n(r,N)$ denotes the number of zeros of $\tilde{s}_N(z)$ in $\bar{D}(0,r)$ counting multiplicity, then the hole probability $P_{0,1}(r,N)$ is just the first term of a sequence of probabilities

$$P_{k,1}(r,N) = \gamma_N\{n(r,N) \le k\}, \;\; k \ge 0.$$

We call $P_{k,1}(r,N)$ a generalized hole probability because compared with the large degree or total number of zeros in $\mathbb{C}$ of the polynomial $\tilde{s}_N$, any finite number k is negligible. It is a status of almost having no zero in $D(0,r)$. And by Theorem 1.6, it turns out that the generalized hole probabilities are numerically almost equal to the regular one.

Proof of Theorem 1.6. (4.21) implies that for all $\eta > 0$,

$$\gamma_N\Big\{\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\, d\sigma_r(u) > \frac{N}{2}\log(1+r^2) + \eta N\Big\} \le e^{-e^{\eta N}} \text{ for } N \gg 1. \tag{7.1}$$

We follow the notations in §4 except this time $m = 1$ and we take the number of partitions $p = 1$. The corresponding statement of Lemma 5.6 is

$$\gamma_N\Big\{\log\prod_{j=0}^{\lfloor\alpha_0 N\rfloor}|\zeta_j| > \frac{o(N^2)}{\delta^2} + (\lfloor\alpha_0 N\rfloor + 1)\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\, d\sigma_r(u)\Big\} \le e^{-e^N} + e^{-R_{\kappa r,1}(N)},$$

where $\zeta_j = \tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}\frac{j}{\lfloor\alpha_0 N\rfloor+1}})$, $0 \le j \le \lfloor\alpha_0 N\rfloor$. Here we do not need to assume $0 \notin \tilde{s}_N\big(\bar{D}(0,r)\big)$ as in Lemma 5.6: the counterpart of II in (4.19) is

$$II = (\lfloor \alpha_0 N \rfloor + 1) \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \int_H P_r(\kappa r e^{2\pi\sqrt{-1}x}, u) \; dx d\sigma_r(u).$$

Since $m = 1$ and $p = 1$, $H = [0,1] \subset \mathbb{R}$,

$$\begin{aligned} II &= (\lfloor \alpha_0 N \rfloor + 1) \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \int_0^1 P_r(\kappa r e^{2\pi\sqrt{-1}x}, u) \; dx d\sigma_r(u) \\ &= (\lfloor \alpha_0 N \rfloor + 1) \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \; d\sigma_r(u). \end{aligned}$$

Therefore, for all $\eta > 0$ small,

$$\begin{aligned} &\gamma_N \Big\{ \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \; d\sigma_r(u) \leq \frac{N}{2} \log (1 + r^2) - \eta N \Big\} \\ \leq & e^{-e^N} + e^{-R_{\kappa r,1}(N)} + \gamma_N \Big\{ \prod_{j=0}^{\lfloor \alpha_0 N \rfloor} |\zeta_j| \leq \exp \Big\{ \frac{o(N^2)}{\delta^2} + (\lfloor \alpha_0 N \rfloor + 1)[\frac{N}{2} \log (1 + r^2) - \eta N] \Big\} \Big\}. \end{aligned} \tag{7.2}$$

Following the steps (4.29)~(4.31), we can show that

$$\begin{aligned} &\log \gamma_N \Big\{ \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \; d\sigma_r(u) \leq \frac{N}{2} \log (1 + r^2) - \eta N \Big\} \\ \leq & N(\lfloor \alpha_0 N \rfloor + 1)[\log(1 + r^2) - 2\eta] - Q_{\kappa r,1,\alpha_0}(N) - 2\beta_1 \alpha_0^2 N^2 + \frac{o(N^2)}{\delta^2}. \end{aligned}$$

$$Q_{\kappa r,1,\alpha_0}(N) \sim N^2 \int_0^{\alpha_0} E_r(x) \; dx = \tfrac{1}{2}\alpha_0[2 \log \kappa r + 1 - \log \alpha_0] N^2,$$

$$\begin{aligned} \beta_1 &= \int_0^1 x \log [2 \sin(\pi x)] \; dx \\ &= \int_0^1 (x - \tfrac{1}{2}) \log [2 \sin(\pi x)] \; dx + \tfrac{1}{2} \int_0^1 \log [2 \sin(\pi x)] \; dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} x \log [2 \sin \pi (x + \tfrac{1}{2})] \; dx + \tfrac{1}{2} \int_0^1 \log [2 \sin(\pi x)] \; dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} x \log [2 \cos(\pi x)] \; dx + \tfrac{1}{2} \int_0^1 \log [2 \sin(\pi x)] \; dx, \end{aligned}$$

as $\int_{-\frac{1}{2}}^{0} x \log [2 \cos(\pi x)] \; dx$ and $\int_0^{\frac{1}{2}} x \log [2 \cos(\pi x)] \; dx$ both converge and $x \log [2 \cos(\pi x)]$ is odd,

$$\beta_1 = \tfrac{1}{2} \int_0^1 \log [2 \sin(\pi x)] \; dx = \tfrac{1}{2} \int_{\partial D(0,1)} \log |1 - z| \; d\sigma_1(z),$$

which equals 0 as in (6.5). Thus

$$\begin{aligned}&\log\gamma_N\Big\{\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\ d\sigma_r(u)\le\frac{N}{2}\log(1+r^2)-\eta N\Big\}\\&\le-\tfrac{1}{2}\alpha_0[1+2\log(\kappa r)-\log\alpha_0-2\log(1+r^2)+4\eta]N^2+\frac{o(N^2)}{\delta^2}.\end{aligned}\tag{7.3}$$

On the other hand,

$$R_{\kappa r,1}(N)\sim N^2\int_{E_{\kappa r}(x)\ge 0}E_{\kappa r}(x)\ dx.\tag{7.4}$$

Combining (7.2)~(7.4), and letting $\delta\to 0+$, we get

$$\begin{aligned}&\log\gamma_N\Big\{\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\ d\sigma_r(u)\le\frac{N}{2}\log(1+r^2)-\eta N\Big\}\\&\le-\min\Big\{\tfrac{1}{2}\alpha_0[1+2\log r-\log\alpha_0-2\log(1+r^2)+4\eta],\tfrac{1}{2}\alpha_0[1+2\log r-\log\alpha_0]\Big\}N^2+o(N^2)\\&=-\tfrac{1}{2}\alpha_0[1+2\log r-\log\alpha_0-2\log(1+r^2)+4\eta]N^2+o(N^2),\end{aligned}\tag{7.5}$$

for $0<\eta<\frac{1}{2}\log(1+r^2)$. Since

$$\int_{E_r(x)\ge 0}E_r(x)\ dx=\tfrac{1}{2}\alpha_0[1+2\log r-\log\alpha_0]>0\Rightarrow 1+2\log r-\log\alpha_0>0,$$

we can choose $0<\eta<\frac{1}{2}\log(1+r^2)$ close to $\frac{1}{2}\log(1+r^2)$ such that

$$1+2\log r-\log\alpha_0-2\log(1+r^2)+4\eta>0.$$

Therefore (7.5) makes sense. Denote

$$F_\eta(r)=\tfrac{1}{2}\alpha_0[1+2\log r-\log\alpha_0-2\log(1+r^2)+4\eta],$$

so we have

$$\gamma_N\Big\{\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\ d\sigma_r(u)\le\frac{N}{2}\log(1+r^2)-\eta N\Big\}\le e^{-F_\eta(r)N^2+o(N^2)},\ \ 0<\eta<\tfrac{1}{2}\log(1+r^2).\tag{7.6}$$

Let $\rho>1$ to be determined. By discarding a null set, we may assume $\tilde{s}_N(0)\ne 0$, $0\notin\tilde{s}_N\big(\partial D(0,r)\big)$ and $0\notin\tilde{s}_N\big(\partial D(0,\rho^{-1}r)\big)$.

So by Jensen's formula, almost surely,

$$\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\, d\sigma_r(u) = \log|c_0| + \int_0^r \frac{n(t,N)}{t}\, dt, \tag{7.7}$$

$$\int_{\partial D(0,\rho^{-1}r)} \log|\tilde{s}_N(u)|\, d\sigma_{\rho^{-1}r}(u) = \log|c_0| + \int_0^{\rho^{-1}r} \frac{n(t,N)}{t}\, dt. \tag{7.8}$$

Since $n(r,N)$ is increasing with respect to r,

$$\begin{aligned}(7.7) \sim (7.8) \Rightarrow & \int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\, d\sigma_r(u) - \int_{\partial D(0,\rho^{-1}r)} \log|\tilde{s}_N(u)|\, d\sigma_{\rho^{-1}r}(u) \\ &= \int_{\rho^{-1}r}^{r} \frac{n(t,N)}{t}\, dt \le n(r,N)\log\rho,\end{aligned}$$

$$\Rightarrow \ n(r,N) \ge \frac{1}{\log\rho}\Big[\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\, d\sigma_r(u) - \int_{\partial D(0,\rho^{-1}r)} \log|\tilde{s}_N(u)|\, d\sigma_{\rho^{-1}r}(u)\Big]. \tag{7.9}$$

(7.1) $\Rightarrow$ For $\eta_1 > 0$, outside an event of probability at most $e^{-e^{\eta_1 N}}$,

$$\int_{\partial D(0,\rho^{-1}r)} \log|\tilde{s}_N(u)|\, d\sigma_{\rho^{-1}r}(u) \le \frac{N}{2}\log(1+\rho^{-2}r^2) + \eta_1 N, \tag{7.10}$$

(7.6) $\Rightarrow$ For $0 < \eta_2 < \frac{1}{2}\log(1+r^2)$, outside an event of probability at most $e^{-F_{\eta_2}(r)N^2 + o(N^2)}$,

$$\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\, d\sigma_r(u) \ge \frac{N}{2}\log(1+r^2) - \eta_2 N. \tag{7.11}$$

(7.9)~(7.11) $\Rightarrow$ outside an event of probability at most $e^{-e^{\eta_1 N}} + e^{-F_{\eta_2}(r)N^2+o(N^2)}$,

$$n(r,N) \ge \frac{N}{\log\rho}\big[\tfrac{1}{2}\log(1+r^2) - \tfrac{1}{2}\log(1+\rho^{-2}r^2) - (\eta_1+\eta_2)\big].$$

Therefore,

$$\gamma_N\Big\{n(r,N) < \frac{N}{\log\rho}\big[\tfrac{1}{2}\log(1+r^2) - \tfrac{1}{2}\log(1+\rho^{-2}r^2) - (\eta_1+\eta_2)\big]\Big\} \le e^{-e^{\eta_1 N}} + e^{-F_{\eta_2}(r)N^2+o(N^2)},$$

where the right hand side is independent of ρ. We need to choose proper ρ, η_1 and η_2.

For all $\tau > 0$, we set

$$\frac{1}{\log\rho}\big[\tfrac{1}{2}\log(1+r^2) - \tfrac{1}{2}\log(1+\rho^{-2}r^2) - (\eta_1+\eta_2)\big] = \tau,$$

$$\eta_1 + \eta_2 = \eta_\tau(\rho) := \tfrac{1}{2}\log(1+r^2) - \tfrac{1}{2}\log(1+\rho^{-2}r^2) - \tau\log\rho.$$

If $\tau > 0$ is small enough, $\rho_0(\tau) := \sqrt{\tfrac{1-\tau}{\tau}}r > 1$,

$$\eta_\tau'(\rho) = \frac{\rho^{-3}r^2}{1+\rho^{-2}r^2} - \frac{\tau}{\rho} = \frac{(1-\tau)r^2 - \tau\rho^2}{\rho(\rho^2+r^2)} \begin{cases} > 0 & \text{when } 1 < \rho < \rho_0, \\ = 0 & \text{when } \rho = \rho_0, \\ < 0 & \text{when } \rho > \rho_0. \end{cases}$$

$$\begin{aligned}
\Rightarrow (\eta_1+\eta_2)_{\max} &= \eta_\tau(\rho_0(\tau)) \\
&= \tfrac{1}{2}\log(1+r^2) - \tfrac{1}{2}\log\Big(1+\frac{\tau}{1-\tau}\Big) - \tau\big[\tfrac{1}{2}\log(1-\tau) - \tfrac{1}{2}\log\tau + \log r\big] \\
&= \tfrac{1}{2}\log(1+r^2) + \tfrac{1}{2}\log(1-\tau) - \frac{\tau}{2}\log(1-\tau) + \frac{\tau}{2}\log\tau - \tau\log r \\
&= \tfrac{1}{2}\log(1+r^2) + \tfrac{1}{2}\big[\tau\log\tau + (1-\tau)\log(1-\tau) - 2\tau\log r\big].
\end{aligned}$$

For a fixed $r > 0$, we can choose smaller $\tau > 0$ if necessary so that

$$-\tfrac{1}{2}\log(1+r^2) < \tau\log\tau + (1-\tau)\log(1-\tau) - 2\tau\log r < 0.$$

This is possible since

$$\tau\log\tau + (1-\tau)\log(1-\tau) - 2\tau\log r < 0 \text{ if } 0 < \tau < \alpha_0$$

and

$$\lim_{\tau\to 0+}\big[\tau\log\tau + (1-\tau)\log(1-\tau) - 2\tau\log r\big] = 0.$$

Thus for such τ and the corresponding $\rho_0 = \rho_0(\tau)$,

$$\frac{1}{4}\log(1+r^2) < \eta_1 + \eta_2 = \eta_\tau(\rho_0) < \tfrac{1}{2}\log(1+r^2).$$

In this case, for all $0 < \eta_1 < \frac{1}{4}\log(1+r^2)$,

$$0 < \eta_2 = \tfrac{1}{2}\log(1+r^2) + \tfrac{1}{2}\big[\tau\log\tau + (1-\tau)\log(1-\tau) - 2\tau\log r\big] - \eta_1 < \tfrac{1}{2}\log(1+r^2),$$

$$
\begin{aligned}
\gamma_N\{n(r,N)<\tau N\} &= \gamma_N\Big\{n(r,N)<\frac{N}{\log\rho_0}\big[\tfrac{1}{2}\log(1+r^2)-\tfrac{1}{2}\log(1+\rho_0^{-2}r^2)-(\eta_1+\eta_2)\big]\Big\}\\
&\le e^{-e^{\eta_1 N}}+e^{-F_{\eta_2}(r)N^2+o(N^2)}.
\end{aligned}
$$

Fix any $k \ge 0$, when N large enough, $k < \tau N$,

$$
\begin{aligned}
&\exp\{-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)N^2+o(N^2)\} = P_{0,1}(r,N)\le P_{k,1}(r,N)\le\gamma_N\{n(r,N)<\tau N\}\\
\le& e^{-e^{\eta_1 N}}+\exp\bigg\{-\tfrac{1}{2}\alpha_0\Big\{\big(1+2\log r-\log\alpha_0\big)+2\big[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r\big]-4\eta_1\Big\}N^2+o(N^2)\bigg\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)\le\liminf_{N\to\infty}\frac{\log P_{k,1}(r,N)}{N^2}&\le\limsup_{N\to\infty}\frac{\log P_{k,1}(r,N)}{N^2}\\
&\le-\tfrac{1}{2}\alpha_0\Big\{\big(1+2\log r-\log\alpha_0\big)+2\big[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r\big]-4\eta_1\Big\}.
\end{aligned}
$$

Let $\eta_1 \to 0+$ and then $\tau \to 0+$, we have

$$
\lim_{N\to\infty}\frac{\log P_{k,1}(r,N)}{N^2}=-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)\iff\log P_{k,1}(r,N)\sim-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)N^2.
$$

$\square$

Appendix

We now prove the following lemma:

Lemma A.1. *The coefficient of* $g_{m,N}(\xi)$ *in* $\det W_{m,N}(\xi)$ *equals* 1.

Proof. Let $\mathcal{S}_{m,N}$ be the set of bijections from $\Gamma_{m,N}$ to $\Lambda_{m,N}$ and for all $\sigma \in \mathcal{S}_{m,N}$, $J \in \Gamma_{m,N}$, write $\sigma(J) = (\sigma_1(J), \ldots, \sigma_m(J))$. Then

$$\det W_{m,N}(\xi) = \sum_{\sigma \in \mathcal{S}_{m,N}} sgn(\sigma) \prod_{J \in \Gamma_{m,N}} \xi_J^{\sigma(J)} = \sum_{\sigma \in \mathcal{S}_{m,N}} sgn(\sigma) \prod_{J \in \Gamma_{m,N}} \xi_{1,j_1}^{\sigma_1(J)} \cdots \xi_{m,j_m}^{\sigma_m(J)}.$$

To find those $\sigma \in \mathcal{S}_{m,N}$ ending up with $g_{m,N}(\xi)$, it is equivalent to find σ satisfying: for all $1 \le i \le m$,

$$\sum_{J \in \Gamma_{m,N}^{i,k}} \sigma_i(J) = \begin{cases} \binom{k+i-1}{i}\binom{N-k+m-i}{m-i} & 1 \le k \le N, \\ 0 & k = 0, \end{cases} \tag{A.12}$$

where the set $\Gamma_{m,N}^{i,k}$ is defined in (3.7). We are going to prove by induction that

$$\sigma(J) = (j_1, j_2 - j_1, \ldots, j_m - j_{m-1}) \text{ for all } J \in \Gamma_{m,N}. \tag{A.13}$$

First of all, similar to $\Gamma_{m,N}^{i,k}$, we introduce

$$\Lambda_{m,N}^{i,k} = \{(k_1, \ldots, k_m) \in \Lambda_{m,N} : k_1 + \cdots + k_i = k\},$$

then

$$\Lambda_{m,N} = \bigsqcup_{k=0}^{N} \Lambda_{m,N}^{i,k} \quad \text{for all } 1 \le i \le m,$$

and

$$|\Lambda_{m,N}^{i,k}| = \binom{k+i-1}{i-1}\binom{N-k+m-i}{m-i} = |\Gamma_{m,N}^{i,k}|.$$

When $i = 1$, (A.12) shows that for $0 \le k \le N$,

$$\sum_{J \in \Gamma_{m,N}^{1,k}} \sigma_1(J) = k\binom{N-k+m-1}{m-1}, \tag{A.14}$$

where the number of terms in the summation on the left is $|\Gamma_{m,N}^{1,k}| = \binom{N-k+m-1}{m-1} = |\Lambda_{m,N}^{1,k}|$ for all $0 \le k \le N$. Then

$$\begin{aligned}
&k = 0 \text{ in } (A.14) \;\Rightarrow\; \sigma(\Gamma_{m,N}^{1,0}) = \Lambda_{m,N}^{1,0} \;\Rightarrow\; \sigma\Big(\bigsqcup_{k=1}^{N} \Gamma_{m,N}^{1,k}\Big) = \bigsqcup_{k=1}^{N} \Lambda_{m,N}^{1,k},\\
&k = 1 \text{ in } (A.14) \;\Rightarrow\; \sigma(\Gamma_{m,N}^{1,1}) = \Lambda_{m,N}^{1,1} \;\Rightarrow\; \sigma\Big(\bigsqcup_{k=2}^{N} \Gamma_{m,N}^{1,k}\Big) = \bigsqcup_{k=2}^{N} \Lambda_{m,N}^{1,k},\\
&\dots\\
&k = N \text{ in } (A.14) \;\Rightarrow\; \sigma(\Gamma_{m,N}^{1,N}) = \Lambda_{m,N}^{1,N},
\end{aligned}$$

$$\Rightarrow \sigma_1(J) = j_1, \qquad \text{for all } J \in \Gamma_{m,N}.$$

Now assume for some $1 \le i \le m-1$, $(\sigma_1 + \cdots + \sigma_i)(J) = j_i$ for all $J \in \Gamma_{m,N}$. Then for any $1 \le k \le N$,

$$\begin{aligned}
\sum_{J \in \Gamma_{m,N}^{i+1,k}} (\sigma_1 + \cdots + \sigma_{i+1})(J) &= \sum_{J \in \Gamma_{m,N}^{i+1,k}} [j_i + \sigma_{i+1}(J)]\\
&= \sum_{j=0}^{k} j|\Gamma_{m,N}^{i,j} \cap \Gamma_{m,N}^{i+1,k}| + \binom{k+i}{i+1}\binom{N-k+m-i-1}{m-i-1}\\
&= \sum_{j=0}^{k} j\binom{j+i-1}{i-1}\binom{N-k+m-i-1}{m-i-1} + \binom{k+i}{i+1}\binom{N-k+m-i-1}{m-i-1}\\
&= k\binom{k+i}{i}\binom{N-k+m-i-1}{m-i-1},
\end{aligned}$$

where the second term on the second line of the calculation comes from (A.12). And for $k = 0$,

$$\sum_{J \in \Gamma_{m,N}^{i+1,0}} (\sigma_1 + \cdots + \sigma_{i+1})(J) = \sum_{J \in \Gamma_{m,N}^{i+1,0}} [j_i + \sigma_{i+1}(J)] = 0.$$

So for all $0 \le k \le N$,

$$\sum_{J \in \Gamma_{m,N}^{i+1,k}} (\sigma_1 + \cdots + \sigma_{i+1})(J) = k\binom{k+i}{i}\binom{N-k+m-i-1}{m-i-1}, \tag{A.15}$$

where the number of terms in the summation on the left is $|\Gamma_{m,N}^{i+1,k}| = \binom{k+i}{i}\binom{N-k+m-i-1}{m-i-1} = |\Lambda_{m,N}^{i+1,k}|$ for all $0 \le k \le N$.

$$\begin{aligned}
&k = 0 \text{ in } (A.15) \;\Rightarrow\; \sigma(\Gamma_{m,N}^{i+1,0}) = \Lambda_{m,N}^{i+1,0} \;\Rightarrow\; \sigma\Big(\bigsqcup_{k=1}^{N} \Gamma_{m,N}^{i+1,k}\Big) = \bigsqcup_{k=1}^{N} \Lambda_{m,N}^{i+1,k},\\
&k = 1 \text{ in } (A.15) \;\Rightarrow\; \sigma(\Gamma_{m,N}^{i+1,1}) = \Lambda_{m,N}^{i+1,1} \;\Rightarrow\; \sigma\Big(\bigsqcup_{k=2}^{N} \Gamma_{m,N}^{i+1,k}\Big) = \bigsqcup_{k=2}^{N} \Lambda_{m,N}^{i+1,k},\\
&\ldots\\
&k = N \text{ in } (A.15) \;\Rightarrow\; \sigma(\Gamma_{m,N}^{i+1,N}) = \Lambda_{m,N}^{i+1,N},
\end{aligned}$$

$$\Rightarrow (\sigma_1 + \cdots + \sigma_{i+1})(J) = j_{i+1}, \quad \text{for all } J \in \Gamma_{m,N}.$$

Thus, (A.13) is proved. And it is trivial to check that the σ defined in (A.13) satisfies all the equations in (A.12). This means that there is only one $\sigma \in \mathcal{S}_{m,N}$ that ends up with $g_{m,N}(\xi)$, and it turns out to be order preserving. Therefore,

$$\det W_{m,N}(\xi) = g_{m,N}(\xi) + \ldots$$

$\square$

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Curriculum Vitae

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